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Cosmological models in the Nonsymmetric Kaluza–Klein Theory

Abstract. We consider a dynamics of Higgs' field in the framework of cosmological models involving a scalar field Ψ from Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory. The field Ψ plays here a rôle of a quintessence field. We consider phase transition in cosmological models of the second and of the first order due to evolution of Higgs' field. We developed inflationary models including calculation of an amount of inflation. We match some cosmological models, calculating a Hubble parameter and an age of the Universe.

1. Introduction

We develop in the paper cosmological consequences of the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory. The theory has been extensively described in the first point of Ref. [1]. We refer to the book on nonsymmetric field theory for all the details of our theory and we present only cosmological applications of the theory.

Let us give a short description of the theory.

We develop a unification of the Nonsymmetric Gravitational Theory and gauge fields (Yang–Mills' fields) including spontaneous symmetry breaking and the Higgs' mechanism with scalar forces connected to the gravitational constant. The theory is geometric and unifies tensor-scalar gravity with massive gauge theory using a multidimensional manifold in a Jordan–Thiry manner. We use a nonsymmetric version of this theory. The general scheme is the following. We introduce the principal fibre bundle over the base $V = E \times G/G_0$ with the structural group H, where E is a space-time, G is a compact semisimple Lie group, G_0 is its compact subgroup and H is a semisimple compact group. The manifold $M = G/G_0$ has an interpretation as a "vacuum states manifold" if G is broken to G_0 (classical vacuum states). We define on the space-time E, the nonsymmetric tensor $g_{\alpha\beta}$ from N.G.T., which is equivalent to the existence of two geometrical objects

$$\overline{g} = g_{(\alpha\beta)}\overline{\theta}^{\alpha} \otimes \overline{\theta}^{\beta} \tag{1.1}$$

$$\underline{g} = g_{[\alpha\beta]}\overline{\theta}^{\alpha} \wedge \overline{\theta}^{\beta} \tag{1.2}$$

the symmetric tensor $\overline{g}$ and the 2-form $\underline{g}$. Simultaneously we introduce on E two connections from N.G.T. $\overline{W}^{\alpha}_{\beta\gamma}$ and $\overline{\widetilde{\Gamma}}^{\alpha}_{\beta\gamma}$. On the homogeneous space M we define the

nonsymmetric metric tensor

$$g_{\tilde{a}\tilde{b}} = h^0_{\tilde{a}\tilde{b}} + \zeta k^0_{\tilde{a}\tilde{b}} \tag{1.3}$$

where ζ is the dimensionless constant, in a geometric way. Thus we really have the nonsymmetric metric tensor on or $V = E \times G/G_0$.

$$\gamma_{\mathrm{AB}} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & r^2 g_{\tilde{a}\tilde{b}} \end{array}\right) \tag{1.4}$$

r is a parameter which characterizes the size of the manifold $M = G/G_0$. Now on the principal bundle $\underline{P}$ we define the connection ω, which is the 1-form with values in the Lie algebra of H.

After this we introduce the nonsymmetric metric on $\underline{P}$ right-invariant with respect to the action of the group H, introducing scalar field ρ in a Jordan–Thiry manner. The only difference is that now our base space has more dimensions than four. It is $(n_1 + 4)$-dimensional, where $n_1 = \dim(M) = \dim(G) - \dim(G_0)$. In other words, we combine the nonsymmetric tensor γ_{AB} on V with the right-invariant nonsymmetric tensor on the group H using the connection ω and the scalar field ρ. We suppose that the factor ρ depends on a space-time point only. This condition can be abandoned and we consider a more general case where $\rho = \rho(x, y)$, $x \in E$, $y \in M$ resulting in a tower of massive scalar field $\rho_k, k = 1, 2 \ldots$. This is really the Jordan–Thiry theory in the nonsymmetric version but with (n_1+4)-dimensional "space-time". After this we act in the classical manner. We introduce the linear connection which is compatible with this nonsymmetric metric. This connection is the multidimensional analogue of the connection $\widetilde{\overline{\Gamma}}{}^{\alpha}_{\beta\gamma}$ on the space-time E. Simultaneously we introduce the second connection W. The connection W is the multidimensional analogue of the $\overline{W}$-connection from N.G.T. and Einstein's Unified Field Theory. Now we calculate the Moffat–Ricci curvature scalar $R(W)$ for the connection W and we get the following result. $R(W)$ is equal to the sum of the Moffat–Ricci curvature on the space-time E (the gravitational lagrangian in Moffat's theory of gravitation), plus $(n_1 + 4)$-dimensional lagrangian for the Yang–Mills' field from the Nonsymmetric Kaluza–Klein Theory plus the Moffat–Ricci curvature scalar on the homogeneous space G/G_0 and the Moffat–Ricci curvature scalar on the group H plus the lagrangian for the scalar field ρ. The only difference is that our Yang–Mills' field is defined on $(n_1 + 4)$-dimensional "space-time" and the existence of the Moffat–Ricci curvature scalar of the connection on the homogeneous space G/G_0. All of these terms (including $R(\overline{W})$) are multiplied by some factors depending on the scalar field ρ.

This lagrangian depends on the point of $V = E \times G/G_0$ i.e. on the point of the space-time E and on the point of $M = G/G_0$. The curvature scalar on G/G_0 also depends on the point of M.

We now go to the group structure of our theory. We assume G invariance of the connection ω on the principal fibre bundle $\underline{P}$, the so called Wang-condition. According to the Wang-theorem the connection ω decomposes into the connection $\widetilde{\omega}_{\mathrm{E}}$ on the

principal bundle Q over space-time E with structural group G and the multiplet of scalar fields Φ. Due to this decomposition the multidimensional Yang–Mills' lagrangian decomposes into: a 4-dimensional Yang–Mills' lagrangian with the gauge group G from the Nonsymmetric Kaluza–Klein Theory, plus a polynomial of 4th order with respect to the fields Φ, plus a term which is quadratic with respect to the gauge derivative of Φ (the gauge derivative with respect to the connection $\widetilde{\omega}_{\mathrm{E}}$ on a space-time E) plus a new term which is of 2nd order in the Φ, and is linear with respect to the Yang–Mills' field strength. After this we perform the dimensional reduction procedure for the Moffat–Ricci scalar curvature on the manifold $\underline{P}$. We average $R(W)$ with respect to the homogeneous space $M = G/G_0$. In this way we get the lagrangian of our theory. It is the sum of the Moffat–Ricci curvature scalar on E (gravitational lagrangian) plus a Yang–Mills' lagrangian with gauge group G from the Nonsymmetric Kaluza–Klein Theory plus a kinetic term for the scalar field Φ, plus a potential $V(\Phi)$ which is of 4th order with respect to Φ, plus $\mathcal{L}_{\mathrm{int}}$ which describes a nonminimal interaction between the scalar field Φ and the Yang–Mills' field, plus cosmological terms, plus lagrangian for scalar field ρ. All of these terms (including $\overline{R}(\overline{W})$) are multiplied of course by some factors depending on the scalar field ρ. We redefine tensor $g_{\mu\nu}$ and ρ and pass from scalar field ρ to Ψ

$$\rho = e^{-\Psi}. \tag{1.5}$$

After this we get lagrangian which is the sum of gravitational lagrangian, Yang–Mills' lagrangian, Higgs' field lagrangian, interaction term $\mathcal{L}_{\mathrm{int}}$ and lagrangian for scalar field Ψ plus cosmological terms. These terms depend now on the scalar field Ψ. In this way we have in our theory a multiplet of scalar fields (Ψ, Φ). As in the Nonsymmetric-Nonabelian Kaluza–Klein Theory we get a polarization tensor of the Yang–Mills' field induced by the skewsymmetric part of the metric on the space-time and on the group G. We get an additional term in the Yang–Mills' lagrangian induced by the skewsymmetric part of the metric $g_{\alpha\beta}$. We get also $\mathcal{L}_{\mathrm{int}}$, which is absent in the dimensional reduction procedure known up to now. Simultaneously, our potential for the scalar–Higgs' field really differs from the analogous potential. Due to the skewsymmetric part of the metric on G/G_0 and on H it has a more complicated structure. This structure offers two kinds of critical points for the minimum of this potential: Φ^0_{crt} and Φ^1_{crt}. The first is known in the classical, symmetric dimensional reduction procedure and corresponds to the trivial Higgs' field ("pure gauge"). This is the "true" vacuum state of the theory. The second, Φ^1_{crt}, corresponds to a more complex configuration. This is only a local (no absolute) minimum of V. It is a "false" vacuum. The Higgs' field is not a "pure" gauge here. In the first case the unbroken group is always G_0. In the second case, it is in general different and strongly depends on the details of the theory: groups G_0, G, H, tensors ℓ_{ab}, $g_{\tilde{a}\tilde{b}}$ and the constants ζ, ξ. It results in a different spectrum of mass for intermediate bosons. However, the scale of the mass is the same and it is fixed by a constant r ("radius" of the manifold $M = G/G_0$). In the first case $V(\Phi^0_{\mathrm{crt}}) = 0$, in the second case it is, in general, not zero $V(\Phi^1_{\mathrm{crt}}) \neq 0$. Thus, in the first case, the cosmological constant is a sum of the scalar curvature on H and G/G_0, and in the

second case, we should add the value $V(\Phi^1_{\rm crt})$. We proved that using the constant ξ we are able in some cases to make the cosmological constant as small as we want (it is almost zero, maybe exactly zero, from the observational data point of view). Here we can perform the same procedure for the second term in the cosmological constant using the constant ζ. In the first case we are able to make the cosmological constant sufficiently small but this is not possible in general for the second case.

The transition from "false" to "true" vacuum occurs as a second order phase transition. We discuss this transition in context of the first order phase transition in models of the Universe. The interesting point is that there exists an effective scale of masses, which depends on the scalar field Ψ.

Using Palatini variational principle we get an equation for fields in our theory. We find a gravitational equation from N.G.T. with Yang–Mills', Higgs' and scalar sources (for scalar field Ψ) with cosmological terms. This gives us an interpretation of the scalar field Ψ as an effective gravitational constant

$$G_{\rm eff} = G_N e^{-(n+2)\Psi}. \tag{1.6}$$

We get an equation for this scalar field Ψ. Simultaneously we get equations for Yang–Mills' and Higgs' field. We also discuss the change of the effective scale of mass, $m_{\rm eff}$ with a relation to the change of the gravitational constant $G_{\rm eff}$.

In the "true" vacuum case we get that the scalar field Ψ is massive and has Yukawa-type behaviour. In this way the weak equivalence principle is satisfied. In the "false" vacuum case the situation is more complex. It seems that there are possible some scalar forces with infinite range. Thus the two worlds constructed over the "true" vacuum and the "false" vacuum seem to be completely different: with different unbroken groups, different mass spectrum for the broken gauge and Higgs' bosons, different cosmological constants and with different behaviour for the scalar field Ψ. The last point means that in the "false" vacuum case the weak equivalence principle could be violated and the gravitational constant (Newton's constant) would increase in distance between bodies.

We explore Einstein λ-transformation for a connection $W^{\tilde A}{}_{\tilde B}$ ($(m+4)$-dimensional) in order to find an interpretation of $\mathbb{R}_+$ gauge invariance of $\widetilde{W}_\mu$ field. We discuss $\mathbb{R}_+$ and $\mathbf{U}(1)_{\rm F}$ invariance. We decide that $\mathbf{U}(1)_{\rm F}$ invariance from G.U.T. is a local invariance. Due to a geometrical construction we are able to identify $\overline{W}_\mu$ from Moffat's theory of gravitation with the four-potential $\widetilde{A}^{\rm F}_\mu$ corresponding to the $\mathbf{U}(1)_{\rm F}$ group (internal rotations connected to fermion charge). In this way, the fermion number is conserved and plays the role of the second gravitational charge. Due to the Higgs' mechanism $\widetilde{A}^{\rm F}_\mu$ is massive and its strength, $\widetilde{H}^{\rm F}_{\mu\nu}$, is of short range with Yukawa-type behaviour. This has important consequences. The Lorentz-like force term (or Coriolis-like force term) in the equation of motion for a test particle is of short range with Yukawa-type behaviour. The range of this force is smaller than the range of the weak interactions. Thus it is negligible in the equation of motion for a test particle. We discuss the possibility of the cosmological origin of the mass of the scalar field Ψ and geodetic equations on

$\underline{P}$. We consider an infinite tower of scalar fields $\Psi_k(x)$ coming from the expansion of the field $\Psi(x,y)$ on the manifold $M = G/G_0$ into harmonics of the Beltrami–Laplace operator. Due to Friedrichs' theory we can diagonalize an infinite matrix of masses for Ψ_k transforming them into new fields Ψ'_k. The truncation procedure means here to take a zero mass mode Ψ_0 and equal it to Ψ.

We discuss in the paper cosmological models involving field Ψ which plays a rôle of a quintessence field. We find inflationary models of the Universe and discuss a dynamics of the Higgs field. Higgs' field dynamics undergoes a second order phase transition which causes a phase transition in an evolution of the Universe. This ends an inflationary epoch and changes an evolution of the field Ψ. Afterwards we consider the field Ψ as a quintessence field building some cosmological models with a quintessence and even with a K-essence. A dynamics of a Higgs field in several approximations gives us an amount of an inflation. We consider also a fluctuation spectrum of primordial fluctuations caused by a Higgs field and a speed up factor of an evolution. We speculate on a future of the Universe based on our simple model with a special behaviour of a quintessence.

The paper is divided into three sections. In the second section we discuss inflationary models and phase transitions obtained due to dynamics of Higgs' fields. In the third section we present an evolution of Higgs field and quintessential models of the Universe. We consider several approaches to get an amount of inflation. Eventually we calculate an amount of an inflation in a simplified model of Higgs' field evolution and a power spectrum of primordial fluctuations caused by a Higgs field.

2. Inflationary models and phase transitions

According to new observational data [2] concerning distances of type Ia supernovae it seems that we need some kind of "dark energy" which drives the evolution of the Universe. This "dark energy" can be considered as a cosmological constant or more general as cosmological terms in field equations (in the lagrangian). In the case of cosmological model this type of "dark energy"—"vacuum energy" is a cause to accelerate the evolution of the Universe (a scale factor $R(t)$) (see Ref. [3]). The cosmological constant is negligible on the level of the Solar System and on the level of the Galaxy. Moreover it can be important if we consider even nonrelativistic movement of galaxies in a cluster of galaxies (see Ref. [4]). In some papers considered cosmological terms result in changing with time of a cosmological constant (see Ref. [5]). Some of them introduce additional scalar field (or fields) in order to give a field-theoretical description of such an evolution of a cosmological "constant". Those scalar fields are independent in general of the additional scalar fields in inflationary models.

Thus the inflation field (or fields in multicomponent inflation, which can be the same as some of Higgs' fields from G.U.T.-models) can be different from those fields. In particular considering scalar-tensor theories of gravitation results in so called quintessence models (see Ref. [6]). Moreover in such theories there is a natural field-theoretical background for an inconstant "gravitational constant". In such a way this

quintessence field can be used in twofold ways. First as a source of change in space and time of a gravitational constant. Secondly as a source of cosmological terms leading to the model of quintessence and a change in time of a cosmological "constant". In our theory we have a natural occurrence of these phenomena due to the scalar field Ψ (or ρ). Let us consider the lagrangian of our theory paying a special attention to the part involving the scalar field Ψ, i.e.:

$$
\begin{aligned}
L = \overline{R}(\overline{W}) &+ 8\pi G_N\biggl(e^{-(n+2)\Psi}\mathcal{L}_{\mathrm{YM}}(\widetilde{A}) + \frac{e^{-2\Psi}}{4\pi r^2}\mathcal{L}_{\mathrm{kin}}(\overset{\mathrm{gauge}}{\nabla}\Psi) \\
&- \frac{e^{(n-2)\Psi}}{8\pi r^2}V(\Phi) - \frac{e^{(n-2)\Psi}}{2\pi r^4}\mathcal{L}_{\mathrm{int}}(\Phi,\widetilde{A})\biggr) \\
&- 8\pi G_N\mathcal{L}_{\mathrm{scal}}(\Psi) + e^{n\Psi}\left(\frac{e^{2\Psi}\alpha_s^2}{l_{\mathrm{pl}}^2}\widetilde{R}(\widetilde{\Gamma}) + \frac{\widetilde{P}}{r^2}\right),
\end{aligned}
\tag{2.1}
$$

$$
\mathcal{L}_{\mathrm{scal}}(\Psi) = \left(\overline{M}\widetilde{g}^{(\gamma\nu)} + n^2 g^{[\mu\nu]}g_{\delta\mu}\widetilde{g}^{(\delta\gamma)}\right)\Psi_{,\nu}\Psi_{,\gamma}\,.
$$

We put $c = \hbar = 1$.

Now we rewrite the lagrangian (2.1) in the following form.

$$
\begin{aligned}
L = \overline{R}(\overline{W}) &- 8\pi G_N e^{-(n+2)\Psi}L_{\mathrm{matter}} \\
&- 8\pi G_N\mathcal{L}_{\mathrm{scal}}(\Psi) + e^{n\Psi}\left(\frac{e^{2\Psi}}{l_{\mathrm{pl}}^2}\alpha_s^2\widetilde{R}(\widetilde{\Gamma}) + \frac{\widetilde{P}}{r^2}\right),
\end{aligned}
\tag{2.1a}
$$

where in L_{matter} we include all the terms from Eq. (2.1) with Yang-Mills' fields, Higgs' fields, their interactions and coupling to the scalar field Ψ.

The effective gravitational constant is defined by

$$
G_{\mathrm{eff}} = G_N e^{-(n+2)\Psi}
$$

in such a way that the lagrangian of the Yang-Mills field in L_{matter} is without any factor involving scalar field Ψ. If the scalar field Ψ is constant (e.g. $\Psi = 0$) we can redefine all the fields in such a way that we get ordinary (standard) lagrangians for these fields.

Let us consider the situation after a spontaneous symmetry breaking and simplify to the case of $g_{[\mu\nu]} = 0$. We get

$$
L = \overline{\widetilde{R}} - 8\pi G_{\mathrm{eff}}L_{\mathrm{matter}} - 8\pi G_N L_{\mathrm{scal}}(\Psi) + 8\pi G_N U(\Psi) \tag{2.2}
$$

where

$$
L_{\mathrm{scal}} = \overline{M}g^{\gamma\nu}\Psi_{,\nu}\cdot\Psi_{,\gamma}, \quad \overline{M} > 0 \tag{2.3}
$$

$$8\pi G_N U(\Psi) = -\frac{V(\Phi^K_{\text{crt}})}{r^4} l^2_{\text{pl}} e^{(n-2)\Psi} + e^{(n+2)\Psi} \left(\frac{\alpha_s^2 \widetilde{R}(\widetilde{\Gamma})}{l^2_{\text{pl}}} \right) + \frac{\underline{\widetilde{P}}}{r^2} e^{n\Psi}. \tag{2.4}$$

We get the following equations

$$\overline{\widetilde{\widetilde{R}}}_{\mu\nu} - \frac{1}{2}\overline{\widetilde{\widetilde{R}}} g_{\mu\nu} = 8\pi G_{\text{eff}} \overset{\text{matter}}{T}_{\mu\nu} + 8\pi G_N \overset{\text{scal}}{T}_{\mu\nu}, \tag{2.5}$$

where $\overline{\widetilde{\widetilde{R}}}_{\mu\nu}$ and $\overline{\widetilde{\widetilde{R}}}$ are a Ricci tensor and a scalar curvature for a Riemannian geometry generated by $g_{\mu\nu} = g_{(\mu\nu)}$,

$$\overset{\text{matter}}{T}{}^{\mu\nu} = (p+\rho)u^\mu u^\nu - p g^{\mu\nu} \tag{2.6}$$

is an energy-momentum tensor for a matter considered as a radiation plus a dust.

$$8\pi G_N \overset{\text{scal}}{T}_{\mu\nu} = 8\pi G_N \left(\frac{\overline{M}}{2} g_{\mu\nu} \cdot (g^{\alpha\beta} \Psi_{,\alpha} \cdot \Psi_{,\beta}) - \overline{M} \Psi_{,\mu} \cdot \Psi_{,\nu} \right) + g_{\mu\nu} \overline{\lambda}_{cK} \tag{2.7}$$

where

$$\overline{\lambda}_{cK} = 2e^{(n-2)\Psi} \frac{m^4_{\widetilde{A}}}{\alpha_s^4} l^2_{\text{pl}} V(\Phi^K_{\text{crt}}) - e^{(n+2)\Psi} \frac{\alpha_s^2}{2l^2_{\text{pl}}} \widetilde{R}(\widetilde{\Gamma}) - e^{n\Psi} \frac{m^2_{\widetilde{A}}}{2\alpha_s^2} \underline{\widetilde{P}}, \tag{2.8}$$

where $m_{\widetilde{A}}$ is a scale of a mass for massive Yang-Mills' fields (after a spontaneous symmetry breaking for a true vacuum case). It is convenient to write

$$\overline{\lambda}_{cK} = \frac{e^{(n-2)\Psi}}{2} \alpha_K - \frac{e^{n\Psi}}{2} \gamma - \frac{e^{(n+2)\Psi}}{2} \beta \tag{2.9}$$

$$\begin{aligned} \alpha_K &= \alpha_K(\xi, \zeta, m_{\widetilde{A}}, \alpha_s) \\ \beta &= \beta(\xi, \alpha_s, m_{\widetilde{A}}) \\ \gamma &= \gamma(\zeta, m_{\widetilde{A}}, \alpha_s) \end{aligned} \tag{2.9a}$$

$K = 0, 1$ corresponds to true and false vacuum case, i.e.

$$V(\Phi^0_{\text{crt}}) = 0, \ \ V(\Phi^1_{\text{crt}}) \neq 0, \tag{2.10}$$

$$\alpha_0 = 0, \ \ \alpha_1 \neq 0. \tag{2.10a}$$

For a scalar field Ψ we have the following equation:

$$\begin{aligned} 16\pi G_N \overline{M} g^{\alpha\beta} \left(\overline{\widetilde{\widetilde{\nabla}}}_\alpha (\partial_\beta \Psi) \right) + (n-2) e^{(n-2)\Psi} \alpha_K - (n+2) e^{(n+2)\Psi} \beta \\ - n e^{n\Psi} \gamma - (n+2) G_{\text{eff}} T = 0 \end{aligned} \tag{2.11}$$

where $T=\rho-3p$ is a trace of an energy-momentum tensor for a matter field. For we are interested in cosmological models we take for a metric tensor a Robertson-Walker metric:

$$ds^2=dt^2-R^2(t)\left[\frac{dr^2}{1-kr^2}+r^2d\theta^2+r^2\sin^2\theta d\varphi^2\right],\quad k=-1,0,1 \tag{2.12}$$

and we suppose that Ψ,ρ,p are functions of t only.

One gets

$$\begin{aligned}\frac{1}{M_{\rm pl}^2}\ddot{\Psi}&=\frac{3}{M_{\rm pl}^2}H\dot{\Psi}-\left(\frac{n-2}{\overline{M}}\right)\alpha_K e^{(n-2)\Psi}+\frac{(n+2)}{\overline{M}}\beta e^{(n+2)\Psi}\\&\quad+\frac{n}{\overline{M}}e^{n\Psi}\gamma+\frac{(n+2)}{M_{\rm pl}^2}e^{-(n+2)\Psi}\cdot(\rho-3p)\end{aligned} \tag{2.13}$$

$$M_{\rm pl}=\left(\sqrt{8\pi G_N}\right)^{-1}=\frac{m_{\rm pl}}{\sqrt{8\pi}}\,.$$

In this case we get standard equations for a cosmological model adapted to our theory

$$\frac{3\ddot{R}}{R}=-8\pi G_{\rm eff}\left(\frac12(\rho+3p)\right)-8\pi G_N\left(\frac12(\rho_\Psi+3p_\Psi)\right), \tag{2.14}$$

$$R\ddot{R}+2\dot{R}+2k=8\pi G_{\rm eff}\left(\frac12(\rho-p)\right)R^2+8\pi G_N\left(\frac12(\rho_\Psi-p_\Psi)\right)R^2. \tag{2.15}$$

Using Eqs (2.14–15) one easily gets

$$H^2+k=\frac{8\pi G_{\rm eff}}{3}\rho+\frac{8\pi G_N}{3}\rho_\Psi \tag{2.16}$$

where $H=\dfrac{\dot{R}}{R}$ is a Hubble constant

$$\begin{aligned}8\pi G_N\rho_\Psi&=8\pi G_N\frac{\overline{M}}{2}\dot{\Psi}^2+\frac12\overline{\lambda}_{cK}\\8\pi G_N p_\Psi&=8\pi G_N\frac{\overline{M}}{2}\dot{\Psi}^2-\frac12\overline{\lambda}_{cK}.\end{aligned} \tag{2.17}$$

Let us consider a cosmological model for a "false vacuum" case, i.e. without matter and only with a scalar Ψ and a vacuum energy $V(\Phi^1_{\rm crt})\neq 0$. In this case $\rho=p=0$ and we get

$$2\overline{M}\ddot{\Psi}-\frac{6\overline{M}\dot{R}}{R}\dot{\Psi}-M_{\rm pl}^2\frac{d\lambda_{c1}}{d\Psi}=0 \tag{2.18}$$

$$\dot{R}^2+k=\frac13\left(\frac12\,\frac{\overline{M}}{M_{\rm pl}^2}\dot{\Psi}^2+\frac12\lambda_{c1}\right)R^2 \tag{2.19}$$

$$\frac{3\ddot{R}}{R}=-\frac{\overline{M}}{M_{\rm pl}^2}\dot{\Psi}^2+\frac12\lambda_{c1}. \tag{2.20}$$

Let us take

$$\Psi = \Psi_1 = \text{const.} \tag{2.21}$$

Thus we get

$$\frac{d\lambda_{c1}}{d\Psi}(\Psi_1) = 0 \tag{2.22}$$

and

$$(n+2)\beta x_1^4 + n\gamma x_1^2 - (n-2)\alpha_1 = 0 \tag{2.23}$$

$$\dot{R}^2 + k = \frac{1}{6}\lambda_{c1}(x_1)R^2 \tag{2.24}$$

$$\frac{3\ddot{R}}{R} = \frac{1}{2}\lambda_{c1}(x_1) \tag{2.25}$$

$$x_1 = e^{\Psi_1}. \tag{$*$}$$

One gets from Eq. (2.25)

$$\overline{R}(t) = R_0 e^{H_0 t} \tag{2.26}$$

where

$$H_0 = \sqrt{\frac{\lambda_{c1}(x_1)}{6}} \tag{2.27}$$

is Hubble constant (really constant). From Eq. (2.23) we obtain

$$x_1 = \sqrt{\frac{-n\gamma + \sqrt{n^2\gamma^2 + 4(n^2-4)\alpha_1\beta}}{2(n+2)\beta}}. \tag{2.28}$$

For $\alpha_1 > 0$, $\beta > 0$ we get

$$\sqrt{n^2\gamma^2 + 4(n^2-4)\alpha_1\beta} > n|\gamma| \tag{2.29}$$

$$\lambda_{c1}(x_1) = \frac{x_1^{n-2}}{2(n+2)^2\beta}\left[n\gamma^2 - \gamma\sqrt{n^2\gamma^2 + 4(n^2-4)\alpha_1\beta} + 4\alpha_1\beta(n+2)\right] \tag{2.30}$$

$$H_0 = \frac{x_1^{(n-2)/2}}{2(n+2)\sqrt{3\beta}}\left[n\gamma^2 - \gamma\sqrt{n^2\gamma^2 + 4(n^2-4)\alpha_1\beta} + 4\alpha_1\beta(n+2)\right]^{1/2}. \tag{2.31}$$

Thus we get an exponential expansion of the Universe. Using Eq. (2.24) we get also $k = 0$ (i.e. a flatness of a space).

In this way we get de Sitter model of the Universe. This is of course a very special solution to the Eqs (2.18–20) with very special initial conditions

$$\left.\begin{aligned} \overline{R}(0) &= R_0 \\ \frac{d\overline{R}}{dt}(0) &= H_0 R_0 \\ \Psi(0) &= \Psi_1 \\ \frac{d\Psi}{dt}(0) &= 0 \end{aligned}\right\} \tag{2.32}$$

Let us disturb the solution by a small perturbation and examine its stability. Let

$$\begin{aligned}&\Psi=\Psi_1+\varphi, \quad |\varphi|,|\dot{\varphi}|\ll\Psi_1\\ &R=\overline{R}+\delta R, \quad |\delta R|,|\delta\dot{R}|\ll\overline{R}.\end{aligned} \tag{2.33}$$

One gets in a linear approximation

$$2\overline{M}\ddot{\varphi}+6\overline{M}H_0\dot{\varphi}+M_{\rm pl}^2\frac{d^2\lambda_{c1}}{d\Psi^2}(\Psi_1)\varphi=0 \tag{2.34}$$

$$\frac{3\delta\ddot{R}}{\overline{R}}=-\overline{M}\dot{\varphi}^2 \tag{2.35}$$

$$2\dot{\overline{R}}\delta\dot{R}+k=\frac{1}{6}\overline{M}\dot{\varphi}^2(\overline{R}^2+\overline{R}\delta R). \tag{2.36}$$

One gets

$$\varphi=\varphi_0 e^{-\frac{3}{2}H_0 t}\sin\left(\frac{\sqrt{p}}{2}t+\delta\right) \tag{2.37}$$

where (we take for simplicity $M=\dfrac{\overline{M}}{M_{\rm pl}^2}$)

$$p=-\Delta=\frac{1}{2M}\frac{d^2\lambda_{c1}}{d\Psi^2}(\Psi_1)-9H_0^2>0 \tag{2.38}$$

if

$$\begin{aligned}&M<M_0=\frac{4}{3}\\ &\times\frac{n(n+2)\gamma\sqrt{n^2\gamma^2+4(n^2-4)\alpha_1\beta}-4(n+2)^2(n-1)\alpha_1\beta-n^2(n+2)\gamma^2}{n\gamma^2+4(n+2)\alpha_1\beta-\gamma\sqrt{n^2\gamma^2+4(n^2-4)\alpha_1\beta}}.\end{aligned} \tag{2.39}$$

Thus we get an exponential decay of the solution φ, i.e. a damped oscillation around $\Psi=\Psi_1$. It means the solution $\Psi=\Psi_1$ is stable against small perturbations of initial conditions for Ψ, i.e.

$$\Psi(0)=\Psi_1+\varphi_0\sin\delta \tag{2.40}$$

$$\frac{d\Psi}{dt}(0)=\varphi_0\left(-\tfrac{3}{2}H_0\sin\delta+\cos\delta\right). \tag{2.41}$$

Making φ_0 and δ sufficiently small we can achieve smallness of perturbations of initial conditions. The exponential decay of the solution to Eq. (2.34) can be satisfied also for some different conditions than Eq. (2.39) if we consider aperiodic case. However, we do not discuss it here.

Thus we get

$$0 \le \frac{1}{2}\overline{M}\dot{\varphi}^2 \le a^2 e^{-3H_0 t} \tag{2.42}$$

where a^2 is a constant.

From Eq. (2.35) one gets

$$0 \le \delta\ddot{R} \le a^2 e^{-4H_o t}, \tag{2.43}$$

which means that $\delta\ddot{R} \sim 0$ and $\delta\dot{R} = O(\delta R(0))$. It means $\delta R = \delta R(0) + \delta\dot{R}(0)t$. Moreover from Eq. (2.36) we get that $k = 0$ due to the fact that δR is a linear function of t. In this way we get

$$R(t) \cong \overline{R}(t) + \delta R(0) + \delta\dot{R}(0)t = \left(R_0 + \frac{\delta R(0) + \delta\dot{R}(0)t}{e^{H_0 t}}\right) e^{H_0 t}. \tag{2.44}$$

This means that the small perturbations for initial conditions for R result in a perturbation of R_0 in such a way that R_0 is perturbed by a quickly decaying function. Thus our de Sitter solution is stable under small perturbations and is an attractor for any small perturbed initial data. One can think that this evolution will continue forever. However, we should remember that we are in a "false" vacuum regime and this configuration of Higgs' fields is unstable. There is a stable configuration—a "true" vacuum case for which $\alpha_K = 0$ ($K = 0$). For $\alpha = 0$ the considered de Sitter evolution cannot be continued. Thus we should take under consideration a second order phase transition in the configuration of Higgs' fields from metastable state to stable state, from "false" vacuum to "true" vacuum, from $\Phi^1_{\rm crt}$ to $\Phi^0_{\rm crt}$. In this case the Higgs fields play a rôle of an order parameter. Let us suppose that the scale time of this phase transition is small in comparison to $\dfrac{1}{H_0}$ and let it take place locally. We suppose locally a conservation of a density of an energy. Thus

$$\overset{\rm scal}{T}_{00} = \overset{\rm scal}{T}_{00} + \overset{\rm matter}{T}_{00}. \tag{2.45}$$

We suppose also that the scalar field Ψ will be close to the new equilibrium (new minimum for cosmological terms) and that the matter will consist of a radiation only:

$$\overset{\rm scal}{T}_{00} = \frac{1}{2}\lambda_{c1}(x_1) \tag{2.46a}$$

$$\overset{\rm scal}{T}_{00} + \overset{\rm matter}{T}_{00} = \frac{1}{2}\lambda_{c0}(x_0) + 8\pi G_N \frac{\rho_r}{x_0^{n+2}}. \tag{2.46b}$$

For $x_0 = e^{\Psi_0}$ is a new equilibrium point we have:

$$\frac{d\lambda_{c0}}{d\Psi}(\Psi_0) = 0 \tag{2.47}$$

$$\lambda_{c0} = -e^{(n+2)\Psi}\beta - e^{n\Psi}\gamma \tag{2.48}$$

$$\frac{d\lambda_0}{d\Psi} = -e^{n\Psi}\left((n+2)\beta e^{2\Psi} + n\gamma\right). \tag{2.49}$$

One gets

$$e^{\Psi_0} = x_0 = \sqrt{\frac{n|\gamma|}{(n+2)\beta}} \tag{2.50}$$

and supposing

$$\gamma < 0 \tag{2.51}$$

thus

$$\lambda_{c0}(\Psi_0) = \frac{2|\gamma|^{\frac{n}{2}+1} n^{\frac{n}{2}}}{\beta^{\frac{n}{2}}(n+2)^{\frac{n}{2}+1}}\,. \tag{2.52}$$

From Eqs (2.46ab) one gets

$$\begin{aligned}\rho_r = \frac{1}{16\pi G_N}&\biggl\{\frac{x_1^{n-2}}{(n+2)^2\beta}\Bigl[n\gamma^2 - \gamma\sqrt{n^2\gamma^2 + 4(n^2-4)\alpha_1\beta} + 2\alpha_1\beta(n+4)\Bigr]\\ &- \frac{2|\gamma|^{\frac{n}{2}+1}}{\beta^{\frac{n}{2}}(n+2)^{\frac{n}{2}+1}}\biggr\}\frac{n^{\frac{n+2}{2}}\gamma^{\frac{n+2}{2}}}{(n+2)^{\frac{n+2}{2}}\beta^{\frac{n+2}{2}}}\,.\end{aligned} \tag{2.53}$$

This gives us matching condition for a second order phase transition and simultaneously it is an initial condition for a new epoch of an evolution of the Universe plus a condition $\dot{\Psi}(t_r) = 0$, $R(t_r) = \overline{R}(t_r)$, where t_r is a time for a phase transition to occur. Notice we have simply $\Psi_0 \neq \Psi_1$. Thus we have to do with a discontinuity for a field Ψ. Let us consider Hubble's constants for both phases of the Universe

$$\begin{aligned} H_0^2 &= \frac{\lambda_{c1}(x_1)}{6} \\ H_1^2 &= \frac{\lambda_{c0}(x_0)}{6} = \frac{|\gamma|^{\frac{n}{2}+1} n^{\frac{n}{2}}}{3\beta^{\frac{n}{2}}(n+2)^{\frac{n}{2}+1}} \\ H_0^2 &\neq H_1^2 \end{aligned} \tag{2.54}$$

Summing up we get

$$\begin{aligned} R(t_r) &= \overline{R}(t_r) = R_0 e^{+H_0 t_r} \\ \dot{\Psi}_1(t_r) &= \dot{\Psi}_0(t_r) = 0 \\ \Psi_1 &\neq \Psi_0 \\ H_0^2 &\neq H_1^2 \end{aligned} \tag{2.55}$$

Thus we see that the second order phase transition in the configuration of Higgs' fields results in the first order phase for an evolution of the Universe. We get discontinuity for Hubble constants and values of scalar field before and after phase transition.

Let us calculate deceleration parameters before and after phase transition

$$q = -\frac{R\ddot{R}}{\dot{R}^2} \tag{2.56}$$

$$q = -1 \tag{2.57}$$

before phase transition and
$$q=-1 \tag{2.58}$$
after phase transition.

Let us come back to the Eqs (2.14), (2.16), (2.13). Let us rewrite Eq. (2.13) in the following form
$$\overline{M}\ddot{\Psi}+\frac{6\overline{M}\dot{R}}{R}\dot{\Psi}+M_{\rm pl}^2\frac{d\lambda_{c0}}{d\Psi}=0. \tag{2.59}$$
(Remember we now have to do with a radiation for which the trace of an energy-momentum tensor is zero.)

Let us suppose that $\dot{\Psi}\approx 0$. Thus Eq. (2.59) simplifies
$$\frac{1}{M_{\rm pl}^2}M\ddot{\Psi}+\frac{d\lambda_{c0}}{d\Psi}=0 \tag{2.60}$$
and we get the first integral of motion
$$\frac{\overline{M}}{2M_{\rm pl}^2}\dot{\Psi}^2+\lambda_{c0}=\frac{1}{2}\overline{\delta}=\text{const.} \tag{2.61}$$
$$\overline{M}\dot{\Psi}^2=M_{\rm pl}^2(\overline{\delta}-2\lambda_{c0}). \tag{2.61a}$$
One gets from Eqs (2.14) and (2.16)
$$\frac{3\ddot{R}}{R}=-\frac{1}{M_{\rm pl}^2}\cdot\frac{\rho_r}{e^{(n+2)\Psi}}-\frac{1}{2}\left(\frac{2\overline{M}}{M_{\rm pl}^2}\dot{\Psi}^2-\lambda_{c0}\right) \tag{2.62}$$
$$\dot{R}^2+k=\frac{1}{3M_{\rm pl}^2}\cdot\frac{\rho_r}{e^{(n+2)\Psi}}R^2+\frac{1}{3}\left(\frac{\overline{M}}{2M_{\rm pl}^2}\dot{\Psi}^2+\frac{1}{2}\lambda_{c0}\right)R^2. \tag{2.63}$$
Using Eq (2.61a) we get:
$$\frac{3\ddot{R}}{R}=-\frac{1}{M_{\rm pl}^2}\cdot\frac{\rho_r}{e^{(n+2)\Psi}}-\overline{\delta}+\frac{5}{2}\lambda_{c0} \tag{2.64}$$
$$\dot{R}^2+k=\frac{1}{3M_{\rm pl}^2}\cdot\frac{\rho_r}{e^{(n+2)\Psi}}R^2+\frac{1}{6}(\overline{\delta}-\lambda_{c0})R^2. \tag{2.65}$$
One can derive from Eqs (2.64–65)
$$\frac{d}{dt}(R\dot{R})=-\frac{1}{6}\overline{\delta}R^2+\frac{2}{3}\lambda_{c0}R^2-k \tag{2.66}$$
$$\lambda_{c0}=-e^{n\Psi}(\beta e^{2\Psi}+\gamma). \tag{2.67}$$

Let us take $k = 0$ and $\delta = 0$ and let us change independent and dependent variables in (2.66) using (2.67) and (2.61a). One gets

$$2y^2(y^2-1)\frac{d^2f}{dy^2} + y\left((n+4)y^2-(n+1)y-2\right)\frac{df}{dy} = \frac{4\overline{M}}{3}(y^2-1)f(y) \tag{2.68}$$

where

$$y = \sqrt{\frac{\beta}{|\gamma|}}e^{\Psi} \tag{2.69}$$

and

$$f = R^2. \tag{2.70}$$

It is easy to see that $y^2 > 1$ for

$$\frac{d\Psi}{dt} = \pm\frac{M_{\rm pl}}{\sqrt{\overline{M}}}\cdot\frac{|\gamma|^{\frac{n+2}{4}}}{\beta^{\frac{n}{4}}}\sqrt{y^n(y^2-1)}. \tag{2.71}$$

Moreover according to our assumptions $\dot{\Psi} \approx 0$ and this can be achieved only if $0 < y - 1 < \varepsilon$, where ε is sufficiently small.

Thus for further investigations we take $z = y - 1$, $y = z + 1$. One gets

$$\begin{gathered}2z(z+2)(z+1)^2\frac{d^2f}{dz^2} + (z+1)\left((n+4)z^2+(n+7)z-(n+3)\right)\frac{df}{dz}\\ -\frac{4\overline{M}}{3}z(z+2)f(z) = 0.\end{gathered} \tag{2.72}$$

Let us consider Eqs (2.64–65) supposing $k = 0$ and $\overline{\delta} = 0$.

After eliminating $\ddot{R}$ (via differentiation of Eq. (2.65) with respect to time t) and changing independent and dependent variables one gets:

$$\frac{d}{dy}\left[\left(8\pi G_N\frac{\widetilde{\rho}_r}{y^{n+2}} + \frac{1}{2}y^n(y^2-1)\right)f^2\right] = -\frac{d}{dy}(f^2)y^n(y^2-1) \tag{2.73}$$

where

$$\overline{\rho}_r = \rho_r\frac{|\gamma|^n}{\beta^{n+1}}\,. \tag{2.74}$$

It is convenient for further investigations to consider a parameter

$$\overline{r} = \frac{4\overline{M}}{3}\,. \tag{2.75}$$

In such a way we get

$$2y^2(y^2-1)\frac{d^2f}{dy^2} + y\left((n+4)y^2-(n+1)y-2\right)\frac{df}{dy} - \overline{r}(y^2-1)f(y) \tag{2.76}$$

and

$$2z(z+2)(z+1)^2\frac{d^2f}{dz^2}+(z+1)\big((n+4)z^2+(n+7)z-(n+3)\big)\frac{df}{dz} \\ -\overline{r}z(z+2)f(z)=0. \tag{2.77}$$

One can transform Eq. (2.73) into

$$\frac{d}{dy}\left[\left(8\pi G_N\frac{\widetilde{\rho}_r}{y^{n+2}}+\frac{3}{2}y^n(y^2-1)\right)f^2\right]=y^{n-1}\left[(n+2)y^2-n\right]f^2. \tag{2.78}$$

Changing the independent variable from y to $z=y-1$ one gets:

$$\frac{d}{dz}\left[\left(8\pi G_N\frac{\widetilde{\rho}_r}{(z+1)^{n+2}}+\frac{3}{2}(z+1)^n z(z+2)\right)f^2\right] \\ =(z+1)^{n-1}\left[(n+2)(z+1)^2-n\right]f^2. \tag{2.79}$$

Let us come back to Eq. (2.71) in order to find time-dependence of Ψ. One gets

$$\int\frac{d\Psi}{\sqrt{\beta e^{(n+2)\Psi}-|\gamma|e^{n\Psi}}}=\pm\frac{t-t_1}{\sqrt{\overline{M}}}\,M_{\rm pl} \tag{2.80}$$

or

$$\int\frac{dx}{x\sqrt{\beta x^{n+2}-|\gamma|x^n}}=\pm\frac{t-t_1}{\sqrt{\overline{M}}}\,M_{\rm pl}. \tag{2.80a}$$

If $n=2l$, where l is a natural number, one gets for $\beta>0$

$$\left(\frac{|\gamma|}{\beta}\right)^{-\frac{1}{2}}\left\{\sum_{k=1}^{l}\sum_{p=0}^{k}\frac{\binom{k}{p}(-2)^{k-p}}{2p}\left(\sqrt{\frac{\sqrt{\beta}x-\sqrt{|\gamma|}}{\sqrt{\beta}x+\sqrt{|\gamma|}}}\right)^{2p+1}\right. \\ \left.+\frac{1}{2\sqrt{3}}\ln\left(\frac{\sqrt{\sqrt{\beta}x-|\gamma|}-\sqrt{3(\sqrt{\beta}x+|\gamma|)}}{\sqrt{\sqrt{\beta}x-|\gamma|}+\sqrt{3(\sqrt{\beta}x+|\gamma|)}}\right)\right\}=\pm\frac{(t-t_1)}{\sqrt{\overline{M}}}\,M_{\rm pl}. \tag{2.81}$$

In this case it can be expressed in terms of elementary functions. For a general case of n, β and γ one gets

$$\frac{2x^{-\frac{n}{2}}\,{}_1F_2(\frac{1}{2},-\frac{n}{4},(1-\frac{n}{4});\frac{x^2\beta}{\gamma})}{\sqrt{\gamma}n}=\pm\frac{(t-t_1)}{\sqrt{\overline{M}}}\,M_{\rm pl} \tag{2.82}$$

where ${}_1F_2(a,b,c;z)$ is a hypergeometric function. For a small z (around zero) one finds the following solution to Eq. (2.77)

$$f(z)=C_1z^{\frac{(n+7)}{4}}\left(1-\frac{(n+7)(5n+23)}{8(n+1)}z\right)+C_2\left(1+\frac{\overline{r}}{(n-1)}z^2\right), \tag{2.83}$$

$C_1, C_2 = \text{const}$. Thus

$$R(y) = \left[C_1(y-1)^{\frac{(n+7)}{4}}\left(1 - \frac{(n+7)(5n+23)}{8(n+1)}(y-1)\right)\right.$$
$$\left. + C_2\left(1 + \frac{\overline{r}}{(n-1)}(y-1)^2\right)\right]^{\frac{1}{2}} \tag{2.84}$$

where $y > 1$ (but only a little).

Let us make some simplifications of the formulae taking under consideration that $y - 1$ is very small.

One gets

$$f(y) = C\left(1 + \frac{\overline{r}}{(n-1)}(y-1)^2\right), \quad C > 0 \tag{2.85}$$

$$R(y) = R_1\left(1 + \frac{\overline{r}}{(n-1)}(y-1)^2\right)^{\frac{1}{2}}. \tag{2.86}$$

After some calculations one gets

$$8\pi G_N \widetilde{\rho}_r = y^{2(n+1)}\left(-\frac{2(n+5)}{(n+4)(n-1)}y^4 + \frac{10\overline{r}(n+2)}{(n-1)(n+4)}y^3\right.$$
$$\left. + \frac{(8\overline{r} - (n-1)(n+4))}{2(n-1)(n+4)}y^2 - \frac{2\overline{r}(n+3)}{(n^2-1)}y + \frac{(\overline{r} - n + 1)}{(n-1)}\right) \tag{2.87}$$

(taking into account that $(y-1)$ is very small). Let us make some simplifications in the formula (2.71) for y close to 1. We find:

$$\int \frac{dy}{\sqrt{y-1}} = \pm \frac{|\gamma|^{\frac{n+4}{4}}\sqrt{2}}{\beta^{\frac{n}{4}}\sqrt{\overline{M}}}(t - t_1)M_{\text{pl}} \tag{2.88}$$

and finally

$$y = \frac{|\gamma|^{\frac{n+4}{2}}}{2\overline{M}\beta^{\frac{n}{2}}}(t - t_1)^2 + 1 \tag{2.89}$$

such that $y = 1$ for $t = t_1$.

Let us pass to the minimum value for $\widetilde{\rho}_r$ with respect to y close to 1. Thus we are looking for a minimum of a function

$$\widetilde{\rho} = \frac{1}{8\pi G_N}y^{2(n+1)}\left[-\frac{2(n+5)}{(n+4)(n-1)}y^4 + \frac{10\overline{r}(n+2)}{(n-1)(n+4)}y^3\right.$$
$$\left. + \frac{(8\overline{r} - (n-1)(n+4))}{2(n-1)(n+4)}y^2 - \frac{2\overline{r}(n+3)}{(n^2-1)}y + \frac{(\overline{r} - n + 1)}{(n-1)}\right] = \frac{1}{8\pi G_N}y^{2(n+1)}W_4(y) \tag{2.90}$$

for such a y that is close to 1.

Let us write $y = 1 + \eta$, where η is very small and develop $W_4(1+\eta)$ up to the second order in η. One finds

$$W_4(1+\eta) \simeq V_2(\eta) \tag{2.91}$$

and

$$V_2(\eta) = a\eta^2 + b\eta + c \tag{2.92}$$

where

$$a = \frac{\left(-n^2 + 3n(20\overline{r} - 9) + 4(32\overline{r} - 31)\right)}{2(n+4)(n-1)} \tag{2.93}$$

$$b = \frac{\left(-n^3 - 4n^2(7\overline{r} - 3) + 5n(18\overline{r} - 11) + 22(3\overline{r} - 2)\right)}{(n+4)(n^2-1)} \tag{2.94}$$

$$c = \frac{\left(-3n^3 + 2n^2(9\overline{r} - 8) + n(50\overline{r} - 29) + 4(7\overline{r} - 4)\right)}{2(n+4)(n^2-1)} \tag{2.95}$$

Let us find minimum of $V_2(\eta)$ for $\eta > 0$ such that $V_2(\eta) \geq 0$. For $a < 0$, $b < 0$, $c > 0$ we get

$$V_2(\eta_{\rm min}) = 0 \tag{2.96}$$

and

$$\eta_{\rm min} = \frac{b + \sqrt{b^2 + 4|a|c}}{2|a|}\,. \tag{2.97}$$

In this case

$$(1+\eta_{\rm min})^{2(n+1)} W_4(1+\eta_{\rm min}) > 0 \tag{2.98}$$

and will be close to the real minimum of the function (2.90). Simultaneously

$$\eta_{\rm min} < 1.$$

For

$$\widetilde{\rho}_r = \frac{|\gamma|^n}{\beta^{n+1}}\sigma T^4 \tag{2.99}$$

(where σ is a Stefan-Boltzmann constant) we get $T_{\rm min}$ and we call it a T_d (a decoupling temperature-decoupling of matter and radiation). Thus $y_d = 1 + \eta_{\rm min}$. y_d is reached at a time t_d (a decoupling time)

$$t_d = t_1 + \frac{\eta_{\rm min}\overline{M}\beta^{\frac{n}{2}}}{2|\gamma|^{(\frac{n+4}{2})}}\,M_{\rm pl}. \tag{2.100}$$

From that time (t_d) the evolution of the Universe will be driven by a matter (with good approximation a dust matter) and the scalar field Ψ. The radius of the Universe at $t = t_d$ is equal to

$$R(t_d) = R_1\left(1 + \frac{\overline{r}}{(n-1)}\eta_{\rm min}^2\right)^{\frac{1}{2}}. \tag{2.101}$$

The full field equations (generalized Friedmann equations) are as follows:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_{\rm eff}}{3}\rho_m + \frac{1}{3}\left(\frac{1}{2}\frac{\overline{M}}{M_{\rm pl}^2}\dot{\Psi}^2 + \frac{1}{2}\lambda_{c0}\right) \tag{2.102}$$

$$\frac{3\ddot{R}}{R} = \frac{8\pi G_{\rm eff}}{3}\rho_m - \frac{1}{2}\left(2\frac{\overline{M}}{M_{\rm pl}^2}\dot{\Psi}^2 - \lambda_{c0}\right) \tag{2.103}$$

$$-\frac{2\overline{M}}{M_{\rm pl}^2}\ddot{\Psi} - \frac{6\overline{M}}{M_{\rm pl}^2}\cdot\frac{\dot{R}}{R}\dot{\Psi} - \frac{d\lambda_{c0}}{d\Psi} + (n+2)8\pi\rho_m G_{\rm eff} = 0. \tag{2.104}$$

Before we pass to those equations we answer the question what is a mass of the scalar field Ψ during the de Sitter phases and the radiation era. One gets:

$$-m_1^2 = \frac{M_{\rm pl}^2}{2\overline{M}}\frac{d^2\lambda_{c1}}{d\Psi^2}(\Psi_1) \tag{2.105}$$

$$-m_0^2 = \frac{M_{\rm pl}^2}{2\overline{M}}\frac{d^2\lambda_{c0}}{d\Psi^2}(\Psi_0). \tag{2.106}$$

The second question we answer is an evolution of the Universe during a period from t_r to t_1, i.e. when y is changing from $y = \sqrt{\frac{n}{n+2}}$ to $y = 1$.

This will be simply the de Sitter evolution

$$R(t) = Re^{H_1 t} \tag{2.107}$$

where

$$H_1 = \sqrt{\frac{\lambda_{c0}(x_0)}{6}}\,. \tag{2.108}$$

This will be unstable evolution and will end at $t = t_1$, i.e. for $\dot{\Psi} \simeq 0$ and value of the field Ψ will be changed to the value corresponding to $y = 1$, i.e.

$$\Psi(t_1) = \frac{1}{2}\ln\frac{\beta}{|\gamma|}\,. \tag{2.109}$$

Thus we have to do with two de Sitter phases of the evolution with two different Hubble constants H_1 and H_0. In the third phase (a radiation era) the radius of the Universe is given by the formula

$$R(t) = R_0 e^{H_0 t_r + H_1(t_1 - t_r)}\cdot\left(1 + \frac{|\gamma|^{n+4}}{3\overline{M}(n-1)\beta^n}(t-t_1)^4\right)^{\frac{1}{2}} \tag{2.110}$$

and such an evolution ends for $t = t_d$,

$$R(t_d) = R_0\exp(H_0 t_r + H_1(t_1 - t_r))\cdot\left(1 + \frac{\overline{r}}{(n-1)}\eta_{\rm min}^2\right)^{\frac{1}{2}}. \tag{2.111}$$

In both de Sitter phases the equation of state for the matter described by the scalar field Ψ is the same:

$$\rho_\Psi = -p_\Psi . \tag{2.112}$$

In the radiation era we get

$$p_\Psi = \tfrac{4}{3}\rho_\Psi . \tag{2.113}$$

Now we go to the matter dominated Universe and we change the notation introducing a scalar field Q such that

$$Q = \sqrt{M}\,\frac{\Psi}{M_{\rm pl}}\,. \tag{2.114}$$

In this case we have

$$\rho_Q = \frac{1}{2}\dot{Q}^2 + U(Q) \tag{2.115a}$$

$$p_Q = \frac{1}{2}\dot{Q}^2 - U(Q) \tag{2.115b}$$

where

$$U(Q) = \frac{1}{2}\lambda_{c0}\left(\frac{QM_{\rm pl}}{\sqrt{M}}\right) = -\frac{\beta}{2}e^{aQ} + \frac{1}{2}|\gamma|e^{bQ} \tag{2.116}$$

$$a = \frac{(n+2)}{\sqrt{M}}\,M_{\rm pl},\ \ b = \frac{n}{\sqrt{M}}\,M_{\rm pl}. \tag{2.117}$$

Let us write Eqs (2.102–104) in terms of Q:

$$\ddot{Q} + 3H\dot{Q} + U'(Q) - 8\pi(n+2)G_{\rm eff}\rho_m = 0 \tag{2.118}$$

where

$$G_{\rm eff} = G_N e^{-\frac{(n+2)}{\sqrt{M}}M_{\rm pl}Q} \tag{2.119}$$

$$H^2 = \frac{1}{3}\left(8\pi G_{\rm eff}\rho_m + \frac{1}{2}\dot{Q}^2 + U\right) \tag{2.120}$$

$$\frac{3\ddot{R}}{R} = \frac{1}{2}\left(8\pi G_{\rm eff}\rho_m + 2(\dot{Q}^2 - U)\right). \tag{2.121}$$

Let us define an equation of state

$$W_Q = \frac{p_Q}{\rho_Q} = \frac{\frac{1}{2}\dot{Q}^2 - U}{\frac{1}{2}\dot{Q}^2 + U} \tag{2.122}$$

and a variable

$$x_Q = \frac{1+W_Q}{1-W_Q} = \frac{\frac{1}{2}\dot{Q}^2}{U}, \tag{2.123}$$

which is a ratio of a kinetic energy to a potential energy density for Q. It is interesting to combine (2.118) and (2.120) using (2.122) and (2.123).

Let us define

$$\varOmega_Q = \frac{\frac{1}{2}\dot{Q}^2 + U}{3H^2} \tag{2.124}$$

$$\varOmega_m = \frac{8\pi G_{\text{eff}}\rho_m}{3H^2}. \tag{2.125}$$

One gets

$$\varOmega_m + \varOmega_Q = 1 \tag{2.126}$$

or

$$8\pi G_{\text{eff}}\rho_m = 3(1-\varOmega_Q)H^2. \tag{2.127}$$

We call ρ_m an energy density of a background and for such a matter the equation of state is

$$W_m = \frac{p_m}{\rho_m} = 0. \tag{2.128}$$

It is natural to define

$$\widetilde{\rho}_m = 8\pi G_{\text{eff}}\rho_m \tag{2.129}$$

as an effective energy density of a background with the same equation of a state as (2.128).

Under an assumption of slow roll for Q, i.e.

$$\frac{1}{2}\dot{Q}^2 \ll U(Q) \tag{2.130}$$

one gets

$$W_Q \simeq -1, \tag{2.131}$$

i.e. an equation of state of a cosmological constant evolving in time. In this case $x_Q \simeq 0$. This can give in principle an account for an acceleration of an evolution of the Universe if we suppose those conditions for our contemporary epoch.

3. Evolution of Higgs Field and Quintessential-Cosmological Models

Let us come back to the inflationary era in our model. For our Higgs' field is multicomponent, i.e. it is a multiplet of Higgs' fields, we have to do with so called multicomponent inflation (see Ref. [7]). In this case we can define slow-roll parameter for our model in equations for Higgs' fields.

One gets from the first point of Ref. [1] (see Eq. (5.7.7), p. 385)

$$\frac{d}{dt}\left(L^{d0}_{\tilde{b}}\right)_{av} - 3H\left(L^{d0}_{\tilde{b}}\right)_{av} = -\frac{e^{n\Psi_1}}{2r^2} l^{db} \left\{\frac{\delta V'}{\delta \Phi^b_{\tilde{n}}} g_{\tilde{b}\tilde{n}}\right\}_{av} \tag{3.1}$$

where $L^d_{0\tilde{a}} = L^d_{t\tilde{a}}$ (a time component of $L^d_{\mu\tilde{a}}$) is defined by

$$l_{dc}L^d_{0\tilde{a}} + l_{cd}g_{\tilde{a}\tilde{m}}g^{\tilde{m}\tilde{c}}L^d_{0\tilde{c}} = 2l_{cd}g_{\tilde{a}\tilde{m}}g^{\tilde{m}\tilde{c}}\frac{d}{dt}\left(\Phi^d_{\tilde{c}}\right), \tag{3.2}$$

V', $\dfrac{\delta V'}{\delta \Phi^b_{\tilde{n}}}$ and $\left\{\dfrac{\delta V'}{\delta \Phi^b_{\tilde{n}}}\right\}_{av}$ are defined in [1] (Eq. (5.6.9), (5.6.10), (5.424)).

$$3H^2 = 8\pi G_N\left(\frac{e^{(n-2)\Psi_1}}{r^4}V'(\Phi) - \frac{2e^{-2\Psi_1}}{r^2}\mathcal{L}_{\rm kin}(\dot{\Phi}) + \frac{1}{2}\lambda_{c1}(\Psi_1)\right) \tag{3.3}$$

$$\dot{H} = 8\pi G_N\left(-\frac{2e^{-2\Psi_1}}{r^2}\mathcal{L}_{\rm kin}(\dot{\Phi})\right) \tag{3.4}$$

where H is a Hubble constant and

$$\mathcal{L}_{\rm kin}(\dot{\Phi}) = l_{ab}\left\{g^{\tilde{b}\tilde{n}}L^a_{0\tilde{b}}\frac{d}{dt}\Phi^b_{\tilde{n}}\right\}_{av}. \tag{3.5}$$

Ψ_1 is a constant value for a field Ψ. The slow-roll parameters are defined by

$$\varepsilon = \frac{l_{ab}\left\{g^{\tilde{b}\tilde{n}}L^a_{0\tilde{b}}\frac{d}{dt}\Phi^b_{\tilde{n}}\right\}_{av}}{H^2} \tag{3.6}$$

$$\delta = \frac{l_{ab}\left\{g^{\tilde{b}\tilde{n}}\frac{d}{dt}\left(L^a_{0\tilde{b}}\right)\frac{d}{dt}\left(\Phi^b_{\tilde{n}}\right)\right\}_{av}}{H^2\left(l_{ab}\left\{g^{\tilde{b}\tilde{n}}L^a_{0\tilde{b}}\frac{d}{dt}\left(\Phi^b_{\tilde{n}}\right)\right\}_{av}\right)} \tag{3.7}$$

with the assumptions of the slow roll

$$\varepsilon = O(\eta),\ \ \delta = O(\eta) \tag{3.8}$$

for some small parameter η.

Under slow-roll conditions we have

$$3H\left(L^d_{0\tilde{b}}\right)_{av} + \frac{e^{n\Psi_1}}{2r^2} l^{db} \left(\frac{\delta V'}{\delta \Phi^b_{\tilde{n}}} g_{\tilde{b}\tilde{n}}\right)_{av} \cong 0 \tag{3.9}$$

$$H^2 \cong \frac{8\pi}{3M^2_{\rm pl}} \left\{\frac{e^{(n-2)\Psi_1}}{r^4} V'(\Phi)\right\} + \frac{1}{2}\lambda_{c1}(\Psi_1) \tag{3.10}$$

and with a standard extra assumption

$$\frac{\dot{\delta}}{H} = O(\eta^2). \tag{3.11}$$

We remind that the scalar field Ψ is the more important agent to drive the inflation. However, the full amount of time the inflation takes place depends on an evolution of Higgs' fields (under slow-roll approximation). The evolution must be slow and starting for $\Phi = \Phi^1_{\rm crt}$. It ends at $\Phi = \Phi^0_{\rm crt}$.

Thus we have according to Eq. (2.26)

$$\overline{N} = \ln \frac{R(t_{\rm end})}{R(t_{\rm initial})} = H_0(t_{\rm end} - t_{\rm initial})$$

where

$$\Phi(t_{\rm end}) = \Phi^0_{\rm crt} \tag{3.12}$$

$$\Phi(t_{\rm initial}) = \Phi^1_{\rm crt} \tag{3.13}$$

$$t_{\rm end} = t_r$$

(we omit indices for Φ).

Thus we should solve Eq. (3.9) for initial condition (3.12) and with an assumption $H = H_0$. We have of course a short period of an inflation with $H = H_1$ from $t_{\rm end} = t_r$ to $t = t_1$, i.e.

$$\overline{N}_{\rm tot} = \overline{N} + \overline{N}_1 = H_0(t_r - t_{\rm initial}) + H_1(t_1 - t_r). \tag{3.14}$$

Let us remind to the reader that $\overline{N}$ is called an amount of inflation. Let us consider inflationary fluctuations of Higgs' fields in our theory. We are ignoring gravitational backreaction for the cosmological evolution is driven by the scalar field Ψ. We write the Higgs field $\Phi^a_{\tilde{m}}$ as a sum

$$\Phi^a_{\tilde{m}}(\vec{r}, t) = \Phi^a_{\tilde{m}}(t) + \delta\Phi^a_{\tilde{m}}(\vec{r}, t) \tag{3.15}$$

where $\Phi^a_{\tilde{m}}(t)$ is a solution of Eq. (3.9) with boundary condition (3.12–13) and $\delta\Phi^a_{\tilde{m}}(\vec{r}, t)$ is a small fluctuation which will be written in terms of Fourier modes

$$\delta\Phi^a_{\tilde{m}}(\vec{r}, t) = \sum_{\vec{K}} \delta\Phi^a_{\vec{K}\tilde{m}}(t) e^{i\vec{K}\vec{r}}. \tag{3.16}$$

The full field equation for Higgs' field in the de Sitter background linearized for $\delta\Phi^a_{\vec{K}\tilde{m}}(t)$ can be written

$$\begin{aligned}\frac{d^2}{dt^2}\left(M^d_{\vec{K}\tilde{b}}\right)_{av}+3H_0\frac{d}{dt}\left(M^d_{\vec{K}\tilde{b}}\right)_{av}+\frac{1}{R^2}\vec{K}^2\left(M^d_{\vec{K}}\right)_{av}\\ =-\frac{e^{n\Psi_1}}{2r^2}l^{db}\left\{\frac{\delta^2V'\left(\Phi^a_{\tilde{b}}(t)\right)}{\delta\Phi^b_{\tilde{n}}\delta\Phi^c_{\tilde{m}}}g_{\tilde{b}\tilde{n}}\right\}\delta\Phi^c_{\vec{K}\vec{m}}(t)\end{aligned} \tag{3.17}$$

where

$$l_{dc}M^d_{\vec{K}\tilde{b}}+l_{cd}g_{\tilde{a}\tilde{m}}g^{\tilde{m}\tilde{c}}M^d_{\vec{K}\tilde{c}}=2l_{cd}g_{\tilde{a}\tilde{m}}g^{\tilde{m}\tilde{c}}\delta\Phi^d_{\vec{K}\tilde{c}}. \tag{3.18}$$

Let us change dependent and independent variables.

$$d\tau=\frac{dt}{\overline{R}(t)},\quad \overline{R}(t)=R_0e^{H_0t} \tag{3.19}$$

$$R(\tau)=-\frac{1}{H_0\tau} \tag{3.20}$$

$$-\infty<\tau<0\quad\text{(a conformal time)} \tag{3.21}$$

and

$$\delta\Phi^a_{\vec{K}\tilde{m}}=\frac{\chi^a_{\vec{K}\tilde{m}}}{R}\,. \tag{3.22}$$

Simultaneously we neglect the term with the second derivative of the potential. Eventually we get

$$\frac{d^2}{d\tau^2}\left(\widetilde{M}^d_{\vec{K}\tilde{b}}\right)_{av}-\frac{2}{\tau^2}\frac{d}{d\tau}\left(\widetilde{M}^d_{\vec{K}\tilde{b}}\right)_{av}+\vec{K}^2\left(\widetilde{M}^d_{\vec{K}\tilde{b}}\right)_{av}=0 \tag{3.23}$$

where

$$l_{dc}\widetilde{M}^d_{\vec{K}\tilde{b}}+l_{cd}g_{\tilde{a}\tilde{m}}g^{\tilde{m}\tilde{c}}\widetilde{M}^d_{\vec{K}\tilde{c}}=2l_{cd}g_{\tilde{a}\tilde{m}}g^{\tilde{m}\tilde{c}}\chi^d_{\vec{K}\tilde{c}}. \tag{3.24}$$

Let us notice that $|\vec{K}\tau|$ is the ratio of the proper wavenumber $|K|R^{-1}$ to the Hubble radius $\frac{1}{H_0}$. At early times $|K\tau|\gg 1$, the wavelength is small in comparison to Hubble radius and the mode oscillates as in Minkowski space. However, if τ goes to zero, $|K\tau|$ goes to zero too. The wavelength of the mode is stretched far beyond the Hubble radius. It means the mode freezes. Thus the analyzes of the modes are similar to those in classical symmetric inflationary perturbation theory neglecting the fact that we have to do with multicomponent inflation and that the structure of representation space of Higgs' fields should be taken into account.

However, the form of Eq. (3.23) is exactly the same as in one component symmetric theory if we take $\widetilde{M}^d_{\vec{K}\tilde{a}}$ a field to be considered. The relation (3.24) between $\widetilde{M}^d_{\vec{K}\tilde{a}}$ and $\chi^d_{\vec{K}\tilde{a}}$ is linear and under some simple conditions unambigous.

It is possible to find an exact solution to Eq. (3.23) and Eq. (3.24). One gets

$$\begin{aligned} \chi^a_{\vec{K}\tilde{m}} &= C^a_{1\vec{K}\tilde{m}}\left(\frac{\tau|\vec{K}|\cos(\tau|\vec{K}|)-\sin(\tau|\vec{K}|)}{\tau}\right) \\ &+ C^a_{2\vec{K}\tilde{m}}\left(\frac{\tau|\vec{K}|\sin(\tau|\vec{K}|)+\cos(\tau|\vec{K}|)}{\tau}\right) \end{aligned} \tag{$*$}$$

and

$$\begin{aligned} M^d_{\vec{K}\tilde{b}} &= N^d_{1\vec{K}\tilde{b}}\left(\frac{\tau|\vec{K}|\cos(\tau|\vec{K}|)-\sin(\tau|\vec{K}|)}{\tau}\right) \\ &+ N^d_{2\vec{K}\tilde{b}}\left(\frac{\tau|\vec{K}|\sin(\tau|\vec{K}|)+\cos(\tau|\vec{K}|)}{\tau}\right) \end{aligned} \tag{$**$}$$

where

$$l_{dc}N^d_{i\vec{K}\tilde{b}} + l_{cd}g_{\tilde{a}\tilde{m}}g^{\tilde{m}\tilde{c}}N^d_{i\vec{K}\tilde{c}} = 2l_{cd}g_{\tilde{a}\tilde{m}}g^{\tilde{m}\tilde{c}}C^d_{i\vec{K}\tilde{c}}, \quad i=1,2. \tag{$***$}$$

In this way one obtains

$$\delta\Phi^a_{\vec{K}\tilde{m}} = \frac{1}{R_0}e^{-H_0t}\chi^a_{\vec{K}\tilde{m}}\left(-\frac{e^{-H_0t}}{R_0H_0}\right) \tag{$****$}$$

and using ($*$) one easily gets

$$\begin{aligned} \delta\Phi^a_{\vec{K}\tilde{m}} &= \widetilde{C}^a_{1\vec{K}\tilde{m}}\left(-\frac{|\vec{K}|}{R_0H_0}e^{-H_0t}\cos\left(\frac{|\vec{K}|}{R_0H_0}e^{-H_0t}\right)+\sin\left(\frac{|\vec{K}|}{R_0H_0}e^{-H_0t}\right)\right) \\ &+ \widetilde{C}^a_{2\vec{K}\tilde{m}}\left(\frac{|\vec{K}|}{R_0H_0}\sin\left(\frac{|\vec{K}|}{R_0H_0}e^{-H_0t}\right)+\cos\left(\frac{|\vec{K}|}{R_0H_0}e^{-H_0t}\right)\right) \end{aligned} \tag{V$*$}$$

where

$$\widetilde{C}^a_{i\vec{K}\tilde{m}} = -H_0C^a_{i\vec{K}\tilde{m}}\,. \tag{VI$*$}$$

However, it is not necessary to use a full exact solution (V$*$) to proceed an analysis which we gave before.

Thus the primordial value of $R_{\vec{K}}(t)$ is equal to

$$R_{\vec{K}} = -\left[\frac{H_0}{\frac{d}{dt}(\Phi^d_{\tilde{m}})}\delta\Phi^d_{\vec{K}\tilde{m}}\right]_{\big|t=t^*} \tag{3.25}$$

where $R_{\vec{K}}(t)$ is defined in cosmological perturbation theory as

$$R^{(3)}(t) = 4\frac{\vec{K}^2}{R^2}R_{\vec{K}}(t) \tag{3.26}$$

(see Ref. [8]) and t^* is such that $|\vec{K}| = R(t^*)H_0$, i.e.

$$t^* = \frac{1}{H_0}\ln\left[\frac{|\vec{K}|}{R_0 H_0}\right], \quad |\vec{K}| = K. \tag{3.27}$$

The primordial value is of course time-independent. We suppose that fluctuations of Higgs' fields (the initial values of them) are independent and Gaussian.

Thus we can repeat some classical results concerning a power spectrum of primordial perturbations. One gets

$$\left[\left|\delta\Phi^d_{\vec{K}\tilde{m}}\right|^2\right] = \left(\frac{H_0}{2\pi}\right)^2. \tag{3.28}$$

Thus

$$P_R(K) \cong \left(\frac{H_0}{2\pi}\right)^2 \sum_{\tilde{m},d}\left(\frac{H_0}{\frac{d}{dt}(\Phi^d_{\tilde{m}})}\right)^2_{\big|t=t^*} \tag{3.29}$$

for t^* given by Eq. (3.27).

Taking a specific situation with concrete groups H, G, G_0 and constants ξ, ζ, r we can in principle calculate exactly $P_R(K)$ and a power index n_s. Thus using some specific software packages (i.e. CMBFAST code) we can obtain theoretical curves of CMB (Cosmic Microwave Background) anisotropy including polarization effects.

Let us give the following remark. The condition for slow-roll evolution for Φ can be expressed in terms of the potential V'. If those are impossible to be satisfied for some models (depending on G, G_0, H, G'_0 and parameters ξ, ζ), we can employ a scenario with a tunnel effect from "false" to "true" vacuum and bubbles coalescence. In the last case the time of an inflation (in the first de Sitter phase) depends on the characteristic time of the coalescence.

Let us consider a more general model of the Universe filled with ordinary (dust) matter, a radiation and a quintessence.

From Bianchi identity we have:

$$\frac{d}{dt}\rho_{\text{tot}} + (\rho_{\text{tot}} + p_{\text{tot}})\frac{3\dot{R}}{R} = 0 \tag{3.30}$$

where

$$\rho_{\text{tot}} = \rho_Q + \widetilde{\rho}_m + \widetilde{\rho}_r \tag{3.31}$$

$$p_{\text{tot}} = p_Q + \widetilde{p}_r. \tag{3.32}$$

Let us suppose that the evolution of radiation, ordinary matter (barionic + cold dark matter) and a quintessence are independent. In this way we get independent conservation laws

$$\frac{d}{dt}(R^3\widetilde{\rho}_m) = 0 \tag{3.33}$$

$$\frac{d}{dt}(R^4\widetilde{\rho}_r) = 0 \tag{3.34}$$

and

$$\dot{\rho}_Q + (\rho_Q + p_Q)\frac{3\dot{R}}{R} = 0. \tag{3.35}$$

One gets

$$\widetilde{\rho}_m = \frac{A}{R^3}, \quad A = \text{const.} \tag{3.36}$$

$$\widetilde{\rho}_r = \frac{B}{R^4}, \quad B = \text{const.} \tag{3.37}$$

From Eq. (3.35) we get

$$\dot{\rho}_Q + (1 + W_Q)\rho_Q \cdot \frac{3\dot{R}}{R} = 0. \tag{3.38}$$

Substituting Eqs (3.36–37) into Eqs (2.14) and (2.16), remembering Eqs (2.114) and (2.116–117) one gets

$$\left(\frac{\dot{R}}{R}\right)^2 + k = \frac{1}{3M_{\rm pl}^2}\left(\frac{A}{R} + \frac{B}{R^2}\right) + \frac{1}{3M_{\rm pl}^2}\rho_Q R^2 \tag{3.39}$$

$$\frac{3\ddot{R}}{R} = -\frac{1}{M_{\rm pl}^2}\left(\frac{A}{2R^3} + \frac{2B}{3R^4}\right) - \frac{1}{2M_{\rm pl}^2}(1 + 3W_Q)\rho_Q \tag{3.40}$$

and from Eq. (2.118)

$$\ddot{Q} + 3H\dot{Q} - \frac{(n+2)A}{M_{\rm pl}^2 R^3} = 0. \tag{3.41}$$

However, now Q and $U(Q)$ are redefined in such a way that $\frac{1}{M_{\rm pl}^2}$ factor appears before ρ_Q. This is a simple rescaling. We have of course

$$\widetilde{\rho}_m = e^{-(n+2)\Psi}\rho_m = e^{-aQ}\rho_m \tag{3.42}$$

$$\widetilde{\rho}_r = e^{-(n+2)\Psi}\rho_r = e^{-aQ}\rho_r. \tag{3.43}$$

Thus we get

$$\rho_m = \frac{A}{R^3}e^{aQ} \tag{3.44}$$

$$\rho_r = \frac{B}{R^4}e^{aQ}. \tag{3.45}$$

Let us consider Eqs (3.39–41) and Eq. (3.35). Combining these equations we get the following formula

$$\frac{(n+2)A}{M_{\rm pl}^2 R^3}\sqrt{(1 + W_Q)\rho_Q} = 0. \tag{3.46}$$

The one way to satisfy this equation is to put $W_Q = -1$. It is easy to see that if $A = 0$ then Eq. (3.46) is satisfied trivially. The condition $A = 0$ does not imply $B = 0$ and we can still have $B \neq 0$. For such a solution $Q = Q_0 = \text{const.}$ and $Q_0 = \frac{\sqrt{\overline{M}}}{M_{\rm pl}}\Psi_0$ found earlier

$$e^{\Psi_0} = x_0 = \sqrt{\frac{n|\gamma|}{(n+2)\beta}} = \frac{1}{\alpha_s}\left(\frac{m_{\tilde{A}}}{m_{\rm pl}}\right)\sqrt{\frac{n|\underline{\widetilde{P}}|}{(n+2)\widetilde{R}(\widetilde{\Gamma})}}. \tag{3.47}$$

Thus

$$e^{-aQ_0} = e^{-(n+2)\Psi_0} = \alpha_s^{n+2}\left(\frac{m_{\rm pl}}{m_{\tilde{A}}}\right)^{n+2}\left(\frac{n+2}{n}\right)^{\frac{n+2}{2}}\left(\frac{\widetilde{R}(\widetilde{\Gamma})}{|\underline{\widetilde{P}}|}\right)^{\frac{n+2}{2}} \tag{3.48}$$

or

$$e^{-aQ_0} = \alpha_s^{n+2}\left(\frac{r}{l_{\rm pl}}\right)\left(\frac{n+2}{n}\right)^{\frac{n+2}{2}}\left(\frac{\widetilde{R}(\widetilde{\Gamma})}{|\underline{\widetilde{P}}|}\right)^{\frac{n+2}{2}}. \tag{3.49}$$

Let us put $W_Q = -1$ into Eqs (3.39–40). One gets

$$\dot{R}^2 + k = \frac{1}{3M_{\rm pl}^2}\left(\frac{A}{R} + \frac{B}{R^2} + \rho_Q R^2\right) \tag{3.50}$$

$$\frac{3\ddot{R}}{R} = -\frac{1}{M_{\rm pl}^2}\left(\frac{A}{2R^3} + \frac{2B}{3R^4} + \rho_Q\right). \tag{3.51}$$

From Eq. (3.50) one gets

$$\frac{\pm R\,dR}{\sqrt{\frac{\rho_Q}{3M_{\rm pl}^2}R^4 - kR^2 + \frac{A}{3M_{\rm pl}^2}R + \frac{B}{3M_{\rm pl}^2}}} = dt \tag{3.52}$$

if $A = 0$,

$$\frac{\pm R\,dR}{\sqrt{\frac{\rho_Q}{3M_{\rm pl}^2}R^4 - kR^2 + \frac{B}{3M_{\rm pl}^2}}} = dt. \tag{3.53}$$

Taking $x = R^2$ one gets

$$\int\frac{dx}{\sqrt{\frac{\rho_Q}{3M_{\rm pl}^2}x^2 - kx + \frac{B}{3M_{\rm pl}^2}}} = \pm 2(t - t_0) \tag{3.54}$$

and finally

$$\pm\int\frac{dy}{\sqrt{y^2 - \frac{3kM_{\rm pl}^2}{\sqrt{B\rho_Q}}y + 1}} = \frac{2\sqrt{3}\sqrt{\rho_Q}M_{\rm pl}}{B}(t - t_0) \tag{3.55}$$

where

$$x = \sqrt{\frac{B}{\rho_Q}}y. \tag{3.56}$$

For a flat case $k = 0$ we find:

$$\pm \int \frac{dy}{\sqrt{y^2+1}} = \frac{2\sqrt{3}\sqrt{\rho_Q}M_{\rm pl}}{B}(t-t_0). \tag{3.57}$$

The last formula can be easily integrated and one gets

$$R = \sqrt[4]{\frac{B}{\rho_Q}}\sqrt{\mathrm{sh}\left(\frac{2\sqrt{3}\sqrt{\rho_Q}M_{\rm pl}}{B}(t-t_0)\right)} \tag{3.58}$$

where we take a sign + in the integral on the left hand side of Eq. (3.57).

Let us calculate a Hubble parameter (constant) for our model. One gets

$$H = \frac{\dot{R}}{R} = \sqrt{3}M_{\rm pl}\left(\frac{\sqrt{\rho_Q}}{B}\right)\mathrm{ctgh}\left(\frac{2\sqrt{3}\sqrt{\rho_Q}M_{\rm pl}}{B}(t-t_0)\right). \tag{3.59}$$

For a deceleration parameter one obtains

$$q = -\frac{\ddot{R}R}{R^2} = -\frac{\left(2\,\mathrm{sh}^2(u) - \mathrm{ch}(u)\right)}{\mathrm{ch}(u)\,\mathrm{sh}^{\frac{1}{2}}(u)} \tag{3.60}$$

where

$$u = \frac{2\sqrt{3}\sqrt{\rho_Q}M_{\rm pl}}{B}(t-t_0). \tag{3.61}$$

Thus we get a spatially flat model of a Universe dominated by a radiation which is expanding and an expansion is accelerating.

Let us take $k = 0$ and $B = 0$ in Eq. (3.52). One gets

$$\frac{\pm R\,dR}{\sqrt{\frac{\rho_Q}{3M_{\rm pl}^2}R^4 + \frac{A}{3M_{\rm pl}^2}R}} = dt \tag{3.62}$$

$$R = \sqrt[3]{\frac{A}{\rho_Q}}x, \quad x = R\sqrt[3]{\frac{\rho_Q}{A}}. \tag{3.63}$$

One finally gets

$$\int \frac{x\,dx}{\sqrt{x(x^3+1)}} = \pm\frac{1}{3M_{\rm pl}}\sqrt{\rho_Q}(t-t_0). \tag{3.64}$$

Let us notice that

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{1}{3M_{\rm pl}^2}\left(\frac{A}{R^3} + \rho_Q\right) > \frac{\rho_Q}{M_{\rm pl}} \tag{3.65}$$

$$\frac{\ddot{R}R}{\dot{R}^2} = \frac{(2\rho_Q R^4 - A)R}{2(A+\rho_Q R^3)}\,. \tag{3.66}$$

Thus the model of the Universe is expanding and accelerating.

Let us consider the integral on the left hand side of Eq. (3.64). One gets after some tedious algebra:

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x(x^3+1)}} &= \frac{\sqrt{\sqrt{3}+2}}{12}\left(9-2\sqrt{3}\right)\ln W + \frac{9\sqrt{2}}{52}\sqrt{4+\sqrt{3}}\left(6\sqrt{3}+11\right)\mathit{\Pi} \\ &- \frac{\sqrt{6}}{598}\left(92\sqrt{4\sqrt{3}+3}\left(141\sqrt{3}+272\right) + \left(9\sqrt{3}-15\right)\sqrt{598}\sqrt{16+9\sqrt{3}}\right)F\end{aligned} \tag{3.67}$$

where

$$\begin{aligned} W = &\left(-20 - \frac{9\sqrt{3}}{2} + \frac{(16-7\sqrt{3})}{2}\left(\frac{2x+\sqrt{3}-1}{2x-\sqrt{3}-1}\right)^2\right. \\ &\left. + 2\sqrt{6-\sqrt{3}}\sqrt{\left[\left(\frac{(2x+\sqrt{3}-1)}{(2x-\sqrt{3}-1)}\right)^2 + 7 - 4\sqrt{3}\right]\left[\left(\frac{(2x+\sqrt{3}-1)}{(2x-\sqrt{3}-1)}\right)^2 - 1 + \frac{\sqrt{3}}{2}\right]}\right) \\ &\times \left(\left(\frac{(2x+\sqrt{3}-1)}{(2x-\sqrt{3}-1)}\right)^2 - 1\right)^{-1}, \end{aligned} \tag{3.68}$$

$$\mathit{\Pi} = \mathit{\Pi}\left(\arccos\left(\left(\frac{(2x+(\sqrt{3}-1))}{(2x-(\sqrt{3}+1))}\right)\sqrt{2(2+\sqrt{3})}\right), \left(\sqrt{3}-1\right), \sqrt{\frac{(5-2\sqrt{3})}{13}}\right) \tag{3.69}$$

$$F = F\left(\arccos\left(\left(\frac{(2x+(\sqrt{3}-1))}{(2x-(\sqrt{3}+1))}\right)\sqrt{2(2+\sqrt{3})}\right), \sqrt{\frac{(5-2\sqrt{3})}{13}}\right) \tag{3.70}$$

where $\mathit{\Pi}$ is an elliptic integral of the third kind and F is an elliptic integral of the first

kind

$$F(u,k) = \int_0^u \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \tag{3.71}$$

$$k = \sqrt{\frac{(5-2\sqrt{3})}{13}} \tag{3.72}$$

$$\Pi(u,n,k) = \int_0^u \frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}} \tag{3.73}$$

$$n = \sqrt{3}-1 \tag{3.74}$$

and with the same modulus as $F(u,k)$

$$k^2 = \frac{5-2\sqrt{3}}{13}\,.$$

However, the most important condition for an existence of a physical solution is coming from the first part of a sum in the right hand side of Eq. (3.67).

We should have

$$W > 0. \tag{3.75}$$

In order to analyze condition (3.75) let us write W in the following way

$$W = \frac{-20 - \frac{9\sqrt{3}}{2} + \frac{(16-7\sqrt{3})}{2}y + 2\sqrt{6-\sqrt{3}}\sqrt{(y+7-4\sqrt{3})\left(y-1+\frac{\sqrt{3}}{2}\right)}}{y-1} \tag{3.76}$$

where

$$y = \left(\frac{2x+\sqrt{3}-1}{2x-\sqrt{3}-1}\right)^2. \tag{3.77}$$

Obviously $y \geq 0$.

We have two possibilities:

I

$$y > 1 \tag{3.78}$$

and

$$-20 - \frac{9\sqrt{3}}{2} + \frac{(16-7\sqrt{3})}{2}y + 2\sqrt{6-\sqrt{3}}\sqrt{(y+7-4\sqrt{3})\left(y-1+\frac{\sqrt{3}}{2}\right)} > 0 \tag{3.79}$$

II

$$y < 1 \tag{3.80}$$

and

$$-20-\tfrac{9\sqrt{3}}{2}+\tfrac{(16-7\sqrt{3})}{2}y+2\sqrt{6-\sqrt{3}}\sqrt{(y+7-4\sqrt{3})\left(y-1+\tfrac{\sqrt{3}}{2}\right)}<0. \tag{3.81}$$

The first possibility can easily be solved:

I $$y > 4.0612 \tag{3.82}$$

or for the second possibility

II $$\frac{2-\sqrt{3}}{2} \leq y < 1. \tag{3.83}$$

We have finally

$$0.7916 < x < \frac{\sqrt{3}+1}{2}\,, \tag{3.84}$$

$$\frac{3\sqrt{3}-5}{2} < x < \frac{1}{2}. \tag{3.85}$$

For

$$x > \frac{\sqrt{3}+1}{2} \tag{3.86}$$

we get

$$x < 3.072. \tag{3.87}$$

Let us notice the following: the function W has a finite limit at $x=\frac{\sqrt{3}+1}{2}$. Thus we can remove a singularity at $x=\frac{\sqrt{3}+1}{2}$. In this way we get

$$0.7916 < x < 3.072 \tag{3.88}$$

or

$$\frac{3\sqrt{3}-5}{2} < x < \frac{1}{2}. \tag{3.89}$$

This simply means that the solution exists only for

$$0.7916\sqrt[3]{\frac{A}{\rho_Q}} < R < 3.072\sqrt[3]{\frac{A}{\rho_Q}} \tag{3.90}$$

or

$$\frac{3\sqrt{3}-5}{2}\sqrt[3]{\frac{A}{\rho_Q}} < R < \frac{1}{2}\sqrt[3]{\frac{A}{\rho_Q}}\,. \tag{3.91}$$

Let us calculate a density of a quintessence energy:

$$\begin{aligned}\rho_Q &= M_{\rm pl}^2 U(Q_0) = -\frac{1}{2} e^{n\Psi_0}\left(\beta e^{2\Psi_0} - |\gamma|\right) \\ &= \left(\frac{n}{n+2}\right)^{\frac{n}{2}} \left(\frac{m_{\tilde{A}}}{\alpha_s^{2(n+1)}}\right) \left(\frac{m_{\tilde{A}}}{m_{\rm pl}}\right)^{n} \left(\frac{|\underline{\widetilde{P}}|}{\widetilde{R}(\widetilde{\Gamma})}\right)^{\frac{n}{2}} |\underline{\widetilde{P}}|.\end{aligned} \tag{3.92}$$

An energy density of a matter is equal to

$$\rho_m = \frac{A}{R^3} e^{aQ_0} \simeq \rho_Q e^{aQ_0} \tag{3.93}$$

and

$$\frac{\rho_m}{\rho_Q} \cong e^{aQ_0} = \left(\frac{1}{\alpha_s}\right)^{n+2} \left(\frac{m_{\tilde{A}}}{m_{\rm pl}}\right)^{n+2} \left(\frac{n}{n+2}\right)^{\frac{n+2}{2}} \left(\frac{\widetilde{R}(\widetilde{\Gamma})}{|\underline{\widetilde{P}}|}\right)^{\frac{n+2}{2}}. \tag{3.94}$$

Let us consider the behaviour of the "effective" gravitational constant in our model

$$\begin{aligned}G_{\rm eff} &= G_N e^{-(n+2)\Psi_0} = G_N e^{-aQ_0} = G_N \left(\frac{1}{x_0}\right)^{n+2} \\ &= G_N \alpha_s^{2(n+2)} \left(\frac{m_{\rm pl}}{m_{\tilde{A}}}\right)^{\frac{n+2}{2}} \left(\frac{\widetilde{R}(\widetilde{\Gamma})}{|\underline{\widetilde{P}}|}\right)^{\frac{n+2}{2}} \left(\frac{n+2}{n}\right)^{\frac{n+2}{2}}\end{aligned} \tag{3.95}$$

Thus $G_{\rm eff}$ is a constant. For we are living in that model we should rescale the constant and consider $G_{\rm eff}(Q_0)$ (Eq. (3.95)) a Newton constant. Thus

$$G_N = G_0 \alpha_s^{2(n+2)} \left(\frac{m_{\rm pl}}{m_{\tilde{A}}}\right)^{\frac{n+2}{2}} \left(\frac{n+2}{n}\right)^{\frac{n+2}{2}} \left(\frac{\widetilde{R}(\widetilde{\Gamma})}{|\underline{\widetilde{P}}|}\right)^{\frac{n+2}{2}} \tag{3.96}$$

where G_0 is a different constant responsible for a strength of gravitational interactions in earlier epochs of an evolution of the Universe. It means we should write

$$G_{\rm eff} = \left(G_N \alpha_s^{-2(n+2)} \left(\frac{m_{\tilde{A}}}{m_{\rm pl}}\right)^{\frac{n+2}{2}} \left(\frac{n}{n+2}\right)^{\frac{n+2}{2}} \left(\frac{|\underline{\widetilde{P}}|}{\widetilde{R}(\widetilde{\Gamma})}\right)^{\frac{n+2}{2}}\right) \cdot e^{-(n+2)\Psi}. \tag{3.97}$$

The obtained solution (i.e. (3.67)) is for $W_Q = -1$. However, this is not an attractor of the dynamical equations. This is similar to the tracker solution of Steinhardt (see Ref. [9]). Moreover we cannot apply his method because our equation for a scalar field Ψ (or Q) is different. It contains an additional term with a trace of the energy-momentum tensor (matter + radiation). Only with a radiation filled Universe our equation is the same (if we identify $\widetilde{\rho}_r$ with Steinhardt density of a matter). In this

case we could apply Steinhardt tracker solution method and apply his criterion for a potential $U(Q)$. In our case $U(Q)$ is given by Eq. (2.116)

$$U'(Q) = -\frac{M_{\rm pl}}{2\sqrt{\overline{M}}}\left((n+2)\beta e^{aQ} - n|\gamma|e^{bQ}\right) \tag{3.98}$$

$$U''(Q) = -\frac{M_{\rm pl}^2}{4\sqrt{\overline{M}}}\left((n+2)^2\beta e^{aQ} - n^2|\gamma|e^{bQ}\right). \tag{3.99}$$

The Steinhardt criterion consists in finding

$$\varGamma = \frac{U''U}{(U')^2} = \frac{\left((n+2)^2\beta e^{aQ} - n^2|\gamma|e^{bQ}\right)\left(\beta e^{aQ} - |\gamma|e^{bQ}\right)}{\left((n+2)\beta e^{aQ} - n|\gamma|e^{bQ}\right)^2}. \tag{3.100}$$

To get a tracker solution we need

$$\varGamma \geq 1. \tag{3.101}$$

In our case

$$\varGamma = 1 - \frac{4\beta|\gamma|e^{(a+b)Q}}{\left((n+2)\beta e^{aQ} - n|\gamma|e^{bQ}\right)^2} \tag{3.102}$$

$$\varGamma \simeq 1 - \frac{4|\gamma|}{(n+2)\beta}e^{(b-a)Q} = 1 - \frac{4|\gamma|}{\beta(n+2)}e^{-\frac{2M_{\rm pl}}{\sqrt{\overline{M}}}Q} = 1 - \frac{4|\gamma|}{\beta(n+2)}e^{-2\varPsi}. \tag{3.103}$$

Thus asymptotically we are close to $\varGamma = 1$. Moreover the Steinhardt criterion is only a sufficient condition. Our radiation filled Universe seems to be unstable due to appearing of a matter (a dust matter). Moreover a quintessential period of a Universe model has a severe restrictions of a value of a radius. Thus the solution with $W_Q = -1$ and a matter independently evolving cannot evolve forever.

In order to answer a question what is a further evolution of the Universe we come back to Einstein equations with a scalar field $\varPsi$ and matter sources. Supposing as usual a Robertson-Walker metric and a spatial flatness of the metric we write once again a Bianchi identity

$$\left(\frac{\overset{\rm scal}{T}{}^{\mu\nu}}{M_{\rm pl}^2} + \frac{1}{M_{\rm pl}^2}e^{-(n+2)\varPsi}\overset{\rm matter}{T}{}^{\mu\nu}\right)_{;\nu} = 0 \tag{3.104}$$

where

$$\overset{\rm matter}{T}{}^{\mu\nu} = \rho_m u^\mu u^\nu \tag{3.105}$$

is an energy-momentum tensor for a dust matter. One gets

$$\begin{aligned}\frac{1}{M_{\rm pl}^2}e^{-(n+2)\varPsi}\left(\rho_m + \dot\rho_m + \rho_m\frac{3\dot R}{R} - (n+2)\rho_m\dot\varPsi\right)\\ + \frac{d\lambda_{c0}}{d\varPsi}\dot\varPsi - \frac{\overline{M}}{M_{\rm pl}^2}\dot\varPsi\ddot\varPsi - \frac{\overline{M}}{M_{\rm pl}^2}\dot\varPsi^2\frac{3\dot R}{R} = 0.\end{aligned} \tag{3.106}$$

We make the following ansatz concerning an evolution of a matter density

$$\rho_m = \rho_0 e^{(n+2)\Psi} \tag{3.107}$$
$$\rho_0 = \text{const.} \tag{3.108}$$

In that moment the scalar field Ψ and the matter ρ_m interact nontrivially.

Using Eqs (3.107–108), (3.106) and an equation for a scalar field Ψ one obtains

$$\frac{\rho_0}{M_{\rm pl}^2} - \frac{(n+2)}{M_{\rm pl}^2}\rho_0 + \frac{\rho_0}{M_{\rm pl}^2}\frac{3\dot{R}}{R} = 0. \tag{3.109}$$

Using Eq. (3.109) and Eq. (2.102) one gets

$$\frac{d\Psi}{dt} = \pm\frac{\sqrt{2}M_{\rm pl}}{\sqrt{3\overline{M}}}\sqrt{\left(\frac{n+1}{M_{\rm pl}}\right)^2 - \frac{3\rho_0}{M_{\rm pl}^2} - \frac{3}{2}\lambda_{c0}} \tag{3.110}$$
$$\int\frac{dx}{x\sqrt{\delta + 3\beta x^{n+2} + 3\gamma x^n}} = \pm\frac{M_{\rm pl}}{\sqrt{\overline{M}}}(t-t_0) \tag{3.111}$$

where

$$\delta = 2\left(\frac{n+1}{M_{\rm pl}}\right)^2 - \frac{3\rho_0}{M_{\rm pl}^2} \tag{3.112}$$
$$x = e^{\Psi}. \tag{3.113}$$

Using Eq. (3.111) and Eq. (2.102) one gets

$$\frac{d}{dt}\ln R = \frac{(n+1)}{M_{\rm pl}} \tag{3.114}$$

or

$$R = R_0 e^{(\frac{n+1}{M_{\rm pl}})(t-t_0)}. \tag{3.114a}$$

Let us consider Eq. (3.111) and let us suppose that $x < 1$ (and practically small). In this case we can simplify

$$\int\frac{dx}{x\sqrt{\delta + 3\beta x^{n+2} + 3\gamma x^n}} \simeq \int\frac{dx}{x\sqrt{\delta + 3\gamma x^n}} \tag{3.115}$$

and finally we get

$$\Psi = \frac{1}{2n}\ln\left(\frac{\delta}{3|\gamma|}\right) - \frac{1}{n}\sqrt{\frac{2\delta}{3\overline{M}}}M_{\rm pl}(t-t_0). \tag{3.116}$$

Thus our prediction that $x < 1$ and is small has been justified. Using Eq. (3.116) one easily gets

$$p_\Psi = \frac{\delta M_{\rm pl}^2}{3n^2} + \frac{\sqrt{|\gamma|\delta}}{2\sqrt{3}} \exp\left(-\sqrt{\frac{2\delta}{3\overline{M}}} M_{\rm pl}(t-t_0)\right) \tag{3.117}$$

$$\rho_\Psi = \frac{\delta M_{\rm pl}^2}{3n^2} - \frac{\sqrt{|\gamma|\delta}}{2\sqrt{3}} \exp\left(-\sqrt{\frac{2\delta}{3\overline{M}}} M_{\rm pl}(t-t_0)\right). \tag{3.118}$$

It is easy to see that

$$\frac{p_\Psi}{\rho_\Psi} \underset{t\to\infty}{\longrightarrow} 1. \tag{3.119}$$

Thus in this model we have to do asymptotically (practically very quickly) with a stiff matter equation of state:

$$p = \rho. \tag{3.120}$$

Simultaneously an energy density for a scalar field Ψ is dominated by a kinetic energy which is practically constant and

$$\frac{1}{2}\dot{Q}^2 \cong \frac{\delta M_{\rm pl}^2}{3n^2}. \tag{3.121}$$

In this sense a dark matter generated by Ψ in the model is in some sense K-essence (not a quintessence).

Let us calculate an energy density for a matter

$$\rho_m = \rho_0 \left(\frac{\delta}{3|\gamma|}\right)^{\frac{(n+2)}{2n}} \exp\left(-\frac{(n+2)}{n}\sqrt{\frac{2\delta}{3\overline{M}}} M_{\rm pl}(t-t_0)\right). \tag{3.122}$$

The effective gravitational constant reads

$$G_{\rm eff} = G_N \left(\frac{3|\gamma|}{\delta}\right)^{\frac{(n+2)}{n}} \exp\left(\left(\frac{n+2}{n}\right)\sqrt{\frac{2\delta}{3\overline{M}}} M_{\rm pl}(t-t_0)\right). \tag{3.123}$$

It is easy to write R as a function of Ψ:

$$R = R_0 \exp\left(\frac{n+1}{M_{\rm pl}^2}\sqrt{\overline{M}} \int\limits_{x_0}^{e^\Psi} \frac{dx}{x\sqrt{\delta + 3\beta x^{n+2} + 3\gamma x^n}}\right) \tag{3.124}$$

where

$$x_0 = \left(\frac{3|\gamma|}{\delta}\right)^{2n}. \tag{3.125}$$

Eq. (3.124) can be easily reduced to the Eq. (3.114) using Eq. (3.111).

Thus an energy density of matter goes to zero in an exponential way. The effective gravitational constant is growing exponentially. What does it mean for the future of the Universe? First of all the Universe will be very diluted after some time of such an evolution. Secondly, a relative strength of gravitational interactions will be stronger. This means that if we take substrat particles as cluster of galaxies then in a quite short time any cluster will be a lonely island in the Universe without any communication with other clusters. All the clusters will be beyond a horizon. On the level of a single cluster the strength of gravitational interactions will be stronger and eventually they collapse to form a black hole. In an intermediate time a gravitational dynamics on the level of galaxies' clusters will be governed by a Newtonian dynamics (nonrelativistic) with a gravitational potential corrected by a cosmological constant. The cosmological constant induces a positive pressure (a stiff matter) and will be important up to a moment of sufficiently large gravitational constant. After a sufficiently long time only a classical Newtonian gravity will drive a dynamics of galaxies' clusters. Let us roughly estimate this effect. In order to do this let us suppose that

$$\rho_\Psi = p_\Psi = \frac{\delta M_{\rm pl}^2}{3n^2} = \text{const.} \tag{3.126}$$

In an energy momentum we get

$$\overset{\rm scal}{T}{}^{\mu\nu} = 2 \cdot \frac{\delta M_{\rm pl}^2}{3n^2} u^\mu u^\nu - \frac{\delta M_{\rm pl}^2}{3n^2} g^{\mu\nu}. \tag{3.127}$$

However, we have in the place of Newtonian constant G_N a $G_{\rm eff}$ which is a function of cosmological time and in the equation for gravitational field for $g_{\mu\nu}$ we should put

$$\frac{1}{(M_{\rm pl}^{\rm eff})^2} \overset{\rm scal}{T}{}^{\mu\nu} = \frac{\delta}{3n^2} \left(\frac{M_{\rm pl}}{M_{\rm pl}^{\rm eff}}\right)^2 (2u^\mu u^\nu - g^{\mu\nu}) \tag{3.128}$$

where

$$\begin{aligned} M_{\rm pl}^{\rm eff} = \frac{1}{\sqrt{8\pi G_{\rm eff}}} = M_{\rm pl} &\left(\frac{\alpha_s}{m_{\rm pl} m_{\widetilde{A}}}\right)^{n+2} \left(\frac{2(n+1)^2 - 3\rho_0}{3|\underline{\widetilde{P}}|}\right)^{\frac{n+2}{2}} \\ &\times \exp\left[\left(\frac{1}{2}\left(\frac{n+2}{2}\right)\sqrt{\frac{2\delta}{3\overline{\overline{M}}}}\right) M_{\rm pl}(t-t_0)\right]. \end{aligned} \tag{3.129}$$

Thus

$$\frac{1}{(M_{\rm pl}^{\rm eff})^2} \overset{\rm scal}{T}{}^{\mu\nu} = \widetilde{A} e^{-\kappa(t-t_0)} (2u^\mu u^\nu - g^{\mu\nu}) \tag{3.130}$$

where

$$\kappa = \frac{1}{2}\left(\left(\frac{n+2}{2}\right)\sqrt{\frac{2\delta}{3\overline{M}}}\right) \tag{3.131}$$

$$\widetilde{A} = \frac{3n^2}{\delta}\left(\frac{\alpha_s}{m_{\rm pl}m_{\widetilde{A}}}\right)^{-(n+2)}\left(\frac{2(n+1)^2-3\rho_0}{3|\underline{\widetilde{P}}|}\right)^{-(\frac{n+2}{2})}. \tag{3.132}$$

Consider the following case (the justification will be given below):

$$\frac{1}{(M_{\rm pl}^{\rm eff})^2}\overset{\rm scal}{T}{}^{\mu\nu} = -\widetilde{A}e^{-\kappa(t-t_0)}g^{\mu\nu}. \tag{3.133}$$

Now let us solve the following problem. Let $\overline{\Lambda}$ be a value $\widetilde{A}e^{-\kappa(\bar{t}-t_0)}$ in an established cosmological time $t=\bar{t}$ and the same for $\overline{G}=G_{\rm eff}(\bar{t})$. Let us consider a gravitational field of a point mass M_0 static and spherically symmetric. Consider a metric

$$ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - r^2\left(d\theta^2+\sin^2\theta\, d\theta^2\right) \tag{3.134}$$

in spherical coordinates and Einstein equations

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R = -\overline{\Lambda}g_{\mu\nu} \tag{3.135}$$

for the metric (3.134) where

$$\overline{\Lambda} = \widetilde{A}e^{-\kappa(\bar{t}-t_0)}. \tag{3.136}$$

Via a standard procedure one obtains

$$B(r) = 1-\frac{2\overline{G}M_0}{r}-\frac{\overline{\Lambda}r^2}{3}. \tag{3.137}$$

Thus an effective nonrelativistic Newton-like gravitational potential has a shape

$$V(r) = \frac{\overline{G}M_0}{r}+\frac{\overline{\Lambda}r^2}{6}\,. \tag{3.138}$$

Remembering that $\overline{G}$ is growing exponentially in time and $\overline{\Lambda}$ is decaying in time exponentially we see that the second term is important only for a short time. For a contemporary time

$$\overline{\Lambda} \simeq 10^{-52}\mathrm{m}^{-2} \tag{3.139}$$

according to the Perlmutter data and it could be important on the level of a size of galaxies' cluster. However, in a short time it will be negligible even on that level.

Let us calculate a Schwarzschild radius for a mass M_0 with an effective gravitational constant $G_{\rm eff}$. One gets

$$
\begin{aligned}
r_g^{\rm eff} &= \sqrt{2M_0\overline{G}} = \sqrt{2M_0G_{\rm eff}(\overline{t})} = r_g\left(\frac{m_{\rm pl}m_{\tilde{A}}}{\alpha_s}\right)^{n+2} \\
&\times\left(\frac{3|\widetilde{\underline{P}}|}{2(n+1)^2-3\rho_0}\right)^{\frac{n+2}{2}}\exp\left[\left(\frac{1}{2}\left(\frac{n+2}{2}\right)\sqrt{\frac{2\delta}{3\overline{\overline{M}}}}\right)M_{\rm pl}(t-t_0)\right]
\end{aligned}
\tag{3.140}
$$

Thus $r_g^{\rm eff}$ (calculated for an effective gravitational constant $G_{\rm eff}(\overline{t})$) is growing exponentially. It means that after some time it will be of an order of size of galaxies' cluster. Then galaxies' clusters collapse to form black holes. The time of a collapse is easy to calculate.

$$
r_g^{\rm eff} = r_{\rm cluster} \tag{3.141}
$$

$$
\begin{aligned}
t_{\rm collapse} &= t_0 + (n+2)\sqrt{6\overline{\overline{M}}}\left[\ln\left(\frac{r_{\rm cluster}}{r_g}\right) + (n+2)\ln\left(\frac{m_{\rm pl}m_{\widetilde{A}}}{\alpha_s}\right)\right. \\
&\left.+\left(\frac{n+2}{2}\right)\ln\left(\frac{3|\widetilde{\underline{P}}|}{2(n+1)^2-3\rho_0}\right)\right]\cdot\left[\sqrt{\delta}(n+2)M_{\rm pl}\right]^{-1}
\end{aligned}
\tag{3.142}
$$

where r_g is a gravitational radius of a mass of a galaxies' cluster with $G = G_N$.

Let us calculate a Hubble constant and a deceleration parameter for our model. One gets

$$
H = \frac{\dot{R}}{R} = (n+1)M_{\rm pl} \tag{3.143}
$$

$$
q = -\frac{\ddot{R}R}{\dot{R}^2} = -1. \tag{3.144}
$$

Summing up we get the following scenario of an evolution. The Universe starts in a "false" vacuum state and its stable evolution is driven by a "cosmological constant" formed from $\widetilde{R}(\widetilde{\Gamma})$, $\overline{P}$, $V(\Phi^1_{\rm crt})$ and Ψ_1 being a minimal solution to an effective self-interacting potential for Ψ. This evolution is stable against small perturbation for Ψ and R. The evolution is exponential and a Hubble constant is calculable in terms of our theory (the nonsymmetric Jordan–Thiry theory). This evolution ends at a moment the "false" vacuum state changes into a "true" vacuum state. In that moment an energy of a vacuum is released into radiation (and a matter). The Universe is reheating and a big-bang scenario begins at a very hot stage. Moreover the evolution is still governed by a scalar field Ψ which attains a new minimum of an effective potential. The effective potential is different and a new value of Ψ, a Ψ_0, is different too. The evolution is exponential and a new Hubble constant H_1 is also calculable in terms of

the underlying theory. In this background we have an evolution of a multiplet of scalar fields $\Phi^a_{\tilde{m}}$ from $\Phi^1_{\rm crt}$ to $\Phi^0_{\rm crt}$. This goes to a spectrum of fluctuations calculable in the theory. After that time the field Ψ is slowly changing $\dot{\Psi} \approx 0$ and the temperature of the Universe is going down. The evolution of a radiation is in the second inflationary stage adiabatic and afterwards during next phase ($\dot{\Psi} \simeq 0$) nonadiabatic. The last means there is an energy exchange between radiation and a scalar field Ψ. The total amount of an inflation $\overline{N}_{\rm tot}$ is calculable in the theory. The evolution of a factor $R(t)$ is governed by a simple elementary function of a small rise. After this the radiation condenses into a matter (a dust with zero pressure) and a matter (a dust), a radiation and a scalar field evolve adiabatically without interaction among them. The evolution of a scalar field is governed by a quintessence, i.e. $p_Q = W_Q \rho_Q$, where $W_Q \cong -1$.

The quintessence field Q is a normalized scalar field Ψ. During a radiation era ($\dot{\Psi} \simeq 0$) an equation of state for a scalar field Ψ is different: $p_\Psi = \frac{4}{3}\rho_\Psi$ (similar to stiff matter equation of state). The scale factor $R(t)$ evolves according to some elementary function built from hyperbolic function in radiation + quintessence scenario and according to complicated function formed from the first and the second elliptic integrals (after reversing of those functions). It expands and accelerates. The solution with radiation and quintessence has a restricted behaviour (Eq. (3.90–91)). The energy density of radiation is going to zero. Only a quintessence energy density is constant. The energy of a quintessence is a potential energy dominated. However, the solution cannot be continued up to the end for an interaction between a matter and a scalar field starts to be important. The effective gravitational constant remains constant during the period. Let us remind to the reader that during inflationary epochs the effective gravitational constant was constant. Only during a radiation era it was slightly changing. In order to keep the Universe to evolve forever it is necessary to find a solution with an interaction between a matter and a scalar field Ψ. Using an appropriate ansatz for an evolution of an energy density of a matter we get such a solution. The scale factor $R(t)$ is exponentially expanding.

An energy density of a matter is going to zero and an energy density of a scalar field is approaching (exponentially) a constant. However, now a scalar field Ψ forms a different form of a matter—a stiff matter with an equation of a state $\rho_\Psi \simeq p_\Psi$ ($W_\Psi \simeq 1$). The effective gravitational constant is growing exponentially in time. In contradistinction with the previous period of an evolution the scalar field Ψ forms a matter with a kinetic energy dominance (i.e. K-essence). In the previous period we have to do with a potential energy dominance (i.e. quintessence).

Let us give some remarks on a spatial curvature of a model of the Universe. In the first period (inflationary scenario) the spatial curvature has to be zero and any perturbations of initial conditions cannot change it. In next periods it is natural to suppose that it is not changed. Thus our model of a Universe is spatially flat. In the moment of change of a phase of an evolution we should apply a matching conditions for an energy density and for a scale factor (a radius of a model of a Universe). However, an evolution in such moments undergoes phase transitions for a law of an evolution of a scalar field and a scale factor, which can be considered as a second order or even a first order

phase transition. It is worth to notice that in formulae concerning Hubble constants, deceleration parameters, gravitational constants (effective) meet some parameters from unification theory of fundamental interactions and from cosmology. It seems that we are on a right track to give an account for large number hypothesis by P. Dirac. This demands of course some investigations especially in developing nonsymmetric Kaluza–Klein (Jordan–Thiry) theory. It is also interesting to mention that the scalar field Ψ plays many rôles in our theory. It works as an inflation field during an inflation epoch. (However Higgs' fields $\Phi^{a}_{\tilde{b}}$ play an important rôle in creating fluctuations spectrum.) It plays also a rôle of a quintessence and a K-essence, except its influence on an effective gravitational constant and on an effective scale of masses in unification of fundamental interactions. This field is massive getting masses from several mechanisms. During inflation phases it acquires mass due to spontaneous symmetry breaking, different in both epochs. It gets a mass from a cosmological background. Thus except its cosmological importance the fluctuations of a scalar field around its cosmological value could be detected (in principle) as massive scalar particles of a large mass. In this way an effective gravitational potential in nonrelativistic limit takes a form

$$V(r)=\frac{G_N M_0}{r}\left(1-\frac{\alpha}{r}e^{-(\frac{r}{r_0})}\right)e^{\kappa(t-t_0)}+\frac{\widetilde{\Lambda}r^2}{6}e^{-\kappa(t-t_0)} \tag{3.145}$$

where κ, $\widetilde{\Lambda}$ are given by Eqs (3.131–132), r_0 is a range of massive scalar $\delta\Psi$ (a fluctuation of a scalar field Ψ in a cosmological background). This could be detected as a tiny change of a Newton constant in gravitational law (e.g. the fifth force) on short distances. Let us notice that on short distances and on short time scales an effective gravitational potential reads:

$$V(r)=\frac{G_N M_0}{r}\left(1-\frac{\alpha}{r}\,e^{-(\frac{r}{r_0})}\right). \tag{3.146}$$

On large distances and short time scales

$$V(r)=\frac{G_N M_0}{r}+\frac{\overline{\Lambda}r^2}{6}. \tag{3.147}$$

On short distances and large time scales

$$V(r)=\frac{G_N M_0}{r}\left(1-\frac{\alpha}{r}\,e^{-(\frac{r}{r_0})}\right)e^{\kappa(t-t_0)}. \tag{3.148}$$

On intermediate distances and large time scales

$$V(r)=\frac{G_N M_0}{r}e^{\kappa(t-t_0)}. \tag{3.149}$$

The latest being a cause to collapse a cluster of galaxies into a black hole. It seems also possible to consider an effective coupling constant α_s^{eff} for α_s enters $m_{\tilde{A}}$ and $m_{\tilde{A}}^{\mathrm{eff}}$. However

$$m_{\tilde{A}}^{\mathrm{eff}}=e^{-\Psi}\left(\frac{\alpha_s}{r}\right), \tag{3.150}$$

r is a radius of a manifold G/G_0 (e.g. a radius of a sphere S^2). If we rewrite Eq. (3.150) as

$$\frac{m^{\text{eff}}_{\tilde{A}}}{\alpha^{\text{eff}}_s} = \frac{1}{r}e^{-\Psi}, \tag{3.151}$$

it would be quite difficult to distinguish a drift of $m^{\text{eff}}_{\tilde{A}}$ from α^{eff}_s drift. Only a quotient has a definite dependence on the scalar field Ψ. Especially it is interesting to consider a drift of α_{em} (a fine structure constant) reported from several sources. However, without extending our theory to include fermion (spinor) fields this is impossible. Thus it is necessary to construct a nonsymmetric-geometric version with supersymmetry and supergravity including noncommutative (anticommutative) coordinates to settle the problem. The scalar field Ψ can form an infinite tower of massive scalar fields which could have important astrophysical and high-energy consequences.

Let us consider Eqs (5.2.51) and (5.2.56) from the first point of Ref. [1], p. 318–319, and changing a base in the Lie algebra $\mathfrak{h}$ we get

$$\mu^a_{\hat{\imath}} = \delta^a_{\hat{\imath}} \tag{3.152}$$

and

$$\alpha^c_i = \delta^c_i. \tag{3.153}$$

Simultaneously for $G \subset H$ ($\mathfrak{g} \subset \mathfrak{h}$) we get

$$f^{\hat{\imath}}_{\tilde{a}\tilde{b}} = C^{\hat{\imath}}_{\tilde{a}\tilde{b}} \tag{3.154a}$$

$$f^{\tilde{d}}_{\tilde{a}\tilde{b}} = C^{\tilde{d}}_{\tilde{a}\tilde{b}} \tag{3.154b}$$

where $\tilde{a}, \tilde{b}, \tilde{c}$ refer to the complement $\mathfrak{m}$ in $\mathfrak{g}$ ($\mathfrak{g} = \mathfrak{g}_0 \dot{+} \mathfrak{m}$), $\hat{\imath}, \hat{\jmath}, \hat{k}$ to the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$. Let us suppose a symmetry requirement

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{g}_0. \tag{3.155}$$

From Eq. (3.155) it follows that

$$C^{\tilde{a}}_{\tilde{b}\tilde{c}} = f^{\tilde{a}}_{\tilde{b}\tilde{c}} = 0. \tag{3.156}$$

Eq. (2.5.51) looks like

$$\Phi^c_{\tilde{b}} C^{\tilde{b}}_{\hat{\imath}\tilde{a}} = \Phi^b_{\tilde{a}} C^c_{\hat{\imath}b}. \tag{3.157}$$

Eq. (2.5.55)

$$H^c_{\tilde{a}\tilde{b}} = C^c_{ab}\Phi^a_{\tilde{a}}\Phi^b_{\tilde{b}} - C^c_{\tilde{a}\tilde{b}} - \Phi^c_{\tilde{d}} C^{\tilde{d}}_{\tilde{a}\tilde{b}}. \tag{3.158}$$

Let us come back to Eq. (3.1).

For our inflation driving agent is a scalar field Ψ we do not carry any backreaction of Higgs' fields and we can consider some nonstandard scenarios of an evolution of

those fields. Thus we can consider an evolution which is not a slow-roll evolution. Let us suppose that

$$\frac{d}{dt}\left(\varPhi^a_{\tilde m}\right) \simeq 0. \tag{3.159}$$

In this way we get from (3.159)

$$\frac{d}{dt}\left(L^{d0}_{\tilde b}\right)_{av} + \frac{e^{n\varPsi_1}}{2r^2} l^{db}\left\{\frac{\delta V'}{\delta\varPhi^b_{\tilde n}} g_{\tilde b\tilde n}\right\}. \tag{3.160}$$

The next step in simplification is such that we collapse all the degrees of freedom of Higgs' fields into one

$$\varPhi^a_{\tilde m} = \delta^a_{\tilde m}\varPhi. \tag{3.161}$$

In this way constraints (3.157) are satisfied trivially and Eq. (3.158) is as follows:

$$H^c_{\tilde a\tilde b} = C^c_{\tilde a\tilde b}\left(\varPhi^2 - 1 - \varPhi\right). \tag{3.162}$$

In this way $V = V'$ (no constraints) and

$$V = \frac{A}{\alpha_s^2}\left(\alpha_s^2\varPhi^2 - 1 - \alpha_s\varPhi\right)^2 \tag{3.163}$$

where

$$A = \left[l_{ab}\left(2g^{[\tilde m\tilde n]}g^{[\tilde a\tilde b]}C^c_{\tilde m\tilde n}C^b_{\tilde a\tilde b} - C^b_{\tilde m\tilde n}g^{\tilde a\tilde n}g^{\tilde b\tilde m}\widetilde{L}^a_{\tilde a\tilde b}\right)\right]_{av} \tag{3.164}$$

and

$$L^a_{\tilde a\tilde b} = \widetilde{L}^a_{\tilde a\tilde b}\left(\alpha_s\varPhi^2 - \frac{1}{\alpha_s} - \varPhi\right) \tag{3.165}$$

$$l_{dc}g_{\tilde m\tilde b}g^{\tilde c\tilde m}\widetilde{L}^d_{\tilde c\tilde a} + l_{cd}g_{\tilde a\tilde m}g^{\tilde m\tilde c}\widetilde{L}^d_{\tilde b\tilde c} = 2l_{cd}g_{\tilde a\tilde m}g^{\tilde m\tilde c}C^d_{\tilde b\tilde c}. \tag{3.166}$$

We can get the following identity

$$l_{dc}g^{\tilde c\tilde q}g^{\tilde a\tilde p}\widetilde{L}^d_{\tilde c\tilde a}C^c_{\tilde p\tilde q} + l_{cd}g^{\tilde p\tilde c}g^{\tilde q\tilde b}\widetilde{L}^d_{\tilde b\tilde c}C^c_{\tilde p\tilde q} = 2l_{cd}g^{\tilde p\tilde c}g^{\tilde q\tilde b}C^d_{\tilde b\tilde c}C^c_{\tilde p\tilde q}. \tag{3.167}$$

Using Eq. (3.160) we get the following equation for $\varPhi$:

$$\frac{d^2\varPhi}{dt^2} + \frac{e^{n\varPsi_1}}{2r^2}\frac{\delta V}{d\varPhi} = 0 \tag{3.168}$$

if the following constraints are satisfied

$$\int\sqrt{|\widetilde{g}|}d^nx\left(g_{\tilde q\tilde c}\widetilde{l}^{\tilde p\tilde e} + l^{\tilde e\tilde n}g_{\tilde c\tilde n}\delta^{\tilde p}_{\tilde q}\right) = 2V_2\delta^e_{\tilde c}\delta^{\tilde p}_{\tilde q}. \tag{3.169}$$

Thus from Eq. (3.168) one gets a first integral of motion

$$\frac{\dot{\Phi}^2}{2} + \frac{e^{n\Psi_1}}{2r^2} V = \frac{B}{2} = \text{const.} \tag{3.170}$$

From Eq. (3.170) one gets

$$\int_{\Phi_0}^{\Phi} \frac{d\Phi}{\sqrt{-\frac{e^{n\Psi_1}}{r^2\alpha_s^2} A \left(\alpha_s^2 \Phi^2 - \alpha_s \Phi - 1\right)^2 + B}} = \pm(t - t_0) \tag{3.171}$$

and finally

$$\int_{\varphi_0}^{\varphi} \frac{d\varphi}{\sqrt{(a - \varphi^2 + \varphi + 1)(a + \varphi^2 - \varphi - 1)}} = \pm\sqrt{\frac{Ae^{n\Psi_1}}{r^2}}(t - t_0) \tag{3.172}$$

where

$$a = \alpha_s \sqrt{\frac{Br^2}{Ae^{n\Psi_1}}} \tag{3.173}$$

$$\Phi = \frac{1}{\alpha_s}\varphi. \tag{3.174}$$

In order to find an integral on the left hand side of Eq. (3.172) and its inverse function we use a uniformization theory of algebraic curves. Let

$$f(\varphi) = (a - \varphi^2 + \varphi + 1)(a + \varphi^2 - \varphi - 1) = -\varphi^4 + 2\varphi^3 + \varphi^2 - 2\varphi + (a^2 - 1). \tag{3.175}$$

One easily notices that

$$\varphi_0 = \frac{1 + \sqrt{5 + 4a}}{2} \tag{3.176}$$

is a root of the polynomial $f(\varphi)$ (a real root).

In this way one gets

$$\varphi = \frac{f'(\varphi_0)}{4\left(P(z) - \frac{1}{24}f''(\varphi_0)\right)} + \varphi_0 \tag{3.177}$$

where

$$P(z) = P(z; g_2, g_3) \tag{3.178}$$

is a P Weierstrass function with invariants:

$$g_2 = \frac{5 - 3a^2}{3} \tag{3.179}$$

$$g_3 = \frac{a^2}{12} + \frac{10}{27}. \tag{3.180}$$

One easily finds

$$f'(\varphi_0) = \tfrac{1}{2}(5+3a)\sqrt{5+4a} \tag{3.181}$$
$$f''(\varphi_0) = -2(5+6a) \tag{3.182}$$

and

$$z = \int_{\varphi_0}^{\varphi} [f(x)]^{-\frac{1}{2}}\, dx. \tag{3.183}$$

In this way one gets

$$\Phi(t) = \frac{(5+3a)\sqrt{5+4a}}{8\alpha_s\left(P(z)+\frac{1}{12}(5+6a)\right)} + \frac{1+\sqrt{5+4a}}{2\alpha_s} \tag{3.184}$$

where

$$z = \pm\sqrt{\frac{Ae^{n\Psi_1}}{r^2}}(t-t_0). \tag{3.185}$$

Let us come back to the potential V Eq. (3.163) and let us look for its critical points. One gets

$$\frac{dV}{d\Phi} = \frac{2A}{\alpha_s}\left(\alpha_s^2\Phi^2 - \alpha_s\Phi - 1\right)(2\alpha_s\Phi - 1) \tag{3.186}$$

and from

$$\frac{dV}{d\Phi} = 0 \tag{3.187}$$

easily finds

$$\Phi_1 = \frac{1}{2\alpha_s} \tag{3.188a}$$
$$\Phi_2 = \frac{1-\sqrt{5}}{2\alpha_s} \tag{3.188b}$$
$$\Phi_3 = \frac{1+\sqrt{5}}{2\alpha_s} \tag{3.188c}$$

One obtains

$$V(\Phi_1) = \frac{25A}{16\alpha_s^2} \tag{3.189a}$$
$$V(\Phi_2) = 0 \tag{3.189b}$$
$$V(\Phi_3) = 0. \tag{3.189c}$$

It is easy to see that Φ_1 is a maximum and Φ_2 and Φ_3 are minima of the potential V.

However, in our simplified model of an evolution of Higgs' fields we have not a second (local) minimum. Thus in order to mimic a real situation we start an evolution of a Higgs field from the value

$$\varPhi = 0. \tag{3.190}$$

In this case

$$V(0) = \frac{A}{\alpha_s^2}\,. \tag{3.191}$$

Thus

$$\varPhi(t_{\text{initial}}) = 0 \tag{3.192}$$

$$\varPhi(t_{\text{end}}) = \frac{1+\sqrt{5}}{2\alpha_s}\,. \tag{3.193}$$

Coming back to Eq. (3.172) we get

$$\int_0^{\frac{1+\sqrt{5}}{2}} \frac{d\varphi}{\sqrt{(a-\varphi^2+\varphi+1)(a+\varphi^2-\varphi-1)}} = \sqrt{\frac{Ae^{n\varPsi_1}}{r^2}}(t_{\text{end}}-t_{\text{initial}}). \tag{3.194}$$

Thus the amount of inflation obtained in our evolution is equal to

$$\overline{N}_0 = H_0\sqrt{\frac{r^2}{Ae^{n\varPsi_1}}} \int_0^{\frac{1+\sqrt{5}}{2}} \frac{d\varphi}{\sqrt{(a-\varphi^2+\varphi+1)(a+\varphi^2-\varphi-1)}}\,. \tag{3.195}$$

Supposing simply

$$a < \frac{5}{4} \tag{3.196}$$

one gets

$$\overline{N}_0 = H_0\sqrt{\frac{2r^2}{5Aae^{n\varPsi_1}}} \int_{\varphi_2}^{\varphi_1} \frac{d\theta}{\sqrt{1-\left(\frac{4a+5}{10}\right)\sin^2\theta}} \tag{3.197}$$

where

$$\cos^2\varphi_1 = \frac{1}{5+4a} \tag{3.198}$$

$$\cos^2\varphi_2 = \frac{5}{5+4a} \tag{3.199}$$

or

$$\overline{N}_0 = H_0\sqrt{\frac{2r^2}{5Aae^{n\varPsi_1}}} \int_{\varphi_2}^{\varphi_1} \frac{d\theta}{\sqrt{1-\left(\frac{4a+5}{10}\right)\sin^2\theta}} \tag{3.200}$$

or

$$\overline{N}_0 = \frac{H_0}{\sqrt{5}\alpha_s} \sqrt[4]{\frac{4r^2}{ABe^{n\Psi_1}}} \int\limits_{\varphi_2}^{\varphi_1} \frac{d\theta}{\sqrt{1-\left(\frac{4a+5}{10}\right)\sin^2\theta}}\,. \tag{3.201}$$

In order to get 60-fold inflation

$$\overline{N}_0 \simeq 60 \tag{3.202}$$

we should play with parameters in our theory. Let us notice that in our simple model the constant α_1 equals

$$\alpha_1 = m_{\tilde{A}}^2 \left(\frac{m_{\tilde{A}}}{m_{\rm pl}}\right)^2 \left(\frac{A}{\alpha_s^6}\right) = \frac{1}{r^2}\left(\frac{l_{\rm pl}}{r}\right)^2 \left(\frac{A}{\alpha_s^6}\right) \tag{3.203}$$

(see Eqs (2.8–9a).

Let us consider a slow-roll dynamic of Higgs' field in this simplified model. From Eq. (3.9) one gets

$$\frac{d}{dt}\left(\varPhi_{\tilde{p}}^f\right) = -\frac{e^{n\Psi_1}}{12V_2r^2H_0}\left(l^{fb}g_{\tilde{p}\tilde{n}} + l^{bf}g_{\tilde{n}\tilde{p}}\right)\frac{\delta V'}{\delta\varPhi_{\tilde{n}}^b}\,. \tag{3.204}$$

Using our simplifications from Eqs (3.152–162) and Eqs (3.143–149) one gets

$$\frac{d\varPhi}{dt} = -\frac{e^{n\Psi_1}}{12r^2H_0}\frac{dV}{d\varPhi} \tag{3.205}$$

supposing a condition

$$\delta_{\tilde{p}}^f = l^{f\tilde{n}}g_{\tilde{p}\tilde{n}} + l^{\tilde{n}f}g_{\tilde{n}\tilde{p}} \tag{3.206}$$

or

$$\frac{d\varPhi}{dt} + \frac{e^{n\Psi_1}A}{6\alpha_s r^2H_0}\left(\alpha_s^2\varPhi^2 - \alpha_s\varPhi - 1\right)(2\alpha_s\varPhi - 1) = 0. \tag{3.207}$$

Changing the dependent variable into

$$\varphi = \alpha_s\varPhi \tag{3.208}$$

one gets

$$\frac{d\varphi}{dt} + b\left(\varphi^2 - \varphi - 1\right)(2\varphi - 1) = 0 \tag{3.209}$$

where

$$b = \frac{e^{n\Psi_1}A}{6r^2H_0} > 0. \tag{3.210}$$

From Eq. (3.209) one finds

$$\frac{\varphi^2 - \varphi - 1}{(\varphi - \frac{1}{2})^2} = e^{-5b(t-t_0)} \tag{3.211}$$

and finally

$$\Phi(t)=\frac{1}{2\alpha_s}\left[1\pm\sqrt{\frac{5}{1-e^{-5b(t-t_0)}}}\right]. \tag{3.212}$$

One easily notices that for $(t-t_0)\to\infty$

$$\Phi(t)\to\frac{1}{\alpha_s}\left(\frac{1\pm\sqrt{5}}{2}\right). \tag{3.213}$$

It simply means that a slow-roll inflation never ends.

If we suppose that an inflation starts at $\varphi=3$, i.e. for

$$t_{\text{initial}}=t_0+\frac{1}{5b}\ln\frac{5}{4}\,, \tag{3.214}$$

it is evident that to complete it we need an eternity.

Summing up we get in our simplified model a finite amount of an inflation for a nonstandard dynamic of a Higgs field and an infinite amount of inflation for a slow-roll dynamic. The truth probably is in a middle.

Probably it would be necessary to consider a full equation for a Higgs field in a simplified model

$$\frac{d^2\Phi}{dt^2}-3H_0\frac{d\Phi}{dt}-3bH_0\left(\alpha_s^2\Phi^2-\alpha_s\Phi-1\right)(2\alpha_s\Phi-1)=0 \tag{3.215}$$

or

$$\frac{d^2\varphi}{dt^2}-3H_0\frac{d\varphi}{dt}-3b\alpha_sH_0\left(\varphi^2-\varphi-1\right)(2\varphi-1)=0. \tag{3.215a}$$

Changing an independent variable from t to $\tau=H_0t$ one gets

$$\frac{d^2\varphi}{d\tau^2}-3\frac{d\varphi}{d\tau}-3\frac{b\alpha_s}{H_0}\left(\varphi^2-\varphi-1\right)(2\varphi-1)=0. \tag{3.215b}$$

Let

$$\frac{3b\alpha_s}{H_0}=\overline{a} \tag{3.216}$$

and let us change a dependent variable φ into

$$z=2\varphi-1 \tag{3.217}$$

$$\varphi=\frac{z+1}{2}\,. \tag{3.218}$$

One gets

$$\frac{d^2z}{d\tau^2}-3\frac{dz}{d\tau}-\frac{\overline{a}}{4}z^3-\frac{(15\overline{a})}{2}z=0. \tag{3.219}$$

Now let us take a special value for $\overline{a} = -\frac{4}{15}$ and let us change z into r:

$$z = i\sqrt{5}r. \tag{3.220}$$

One gets

$$\frac{d^2r}{d\tau^2} + 3\frac{dr}{d\tau} - 2r^3 + 2r = 0. \tag{3.221}$$

This equation can be solved in general [10]

$$r(\tau) = iC_1 e^{\tau} \operatorname{sn}(C_1 e^{\tau} + C_2, -1). \tag{3.222}$$

Thus

$$z(\tau) = -\sqrt{5}C_1 e^{\tau} \operatorname{sn}(C_1 e^{\tau} + C_2, -1). \tag{3.223}$$

$\operatorname{sn}(u, -1)$ is a Jacobi elliptic function with a modulus $K^2 = -1$, C_1 and C_2 are constants. In this way

$$\varPhi(t) = \frac{1}{2\alpha_s}\left(1 - \sqrt{5}C_1 e^{H_0 t} \operatorname{sn}(C_1 e^{H_0 t} + C_2, -1)\right). \tag{3.224}$$

However, we are forced to put $\overline{a} = -\frac{4}{15}$. Let us remind that

$$\overline{a} = \frac{e^{n\varPsi_1}}{2r^2 H_0^2} A \tag{3.225}$$

and we are supposing $A > 0$. However (see Eq. (3.164) for a definition of A), it is possible to consider $A < 0$ only for a pleasure to play. Thus we use formula (3.224) to consider a dynamics of a Higgs field. Let us start an inflation for $t = 0$ with $\varPhi = 0$.

$$\varPhi(0) = 0. \tag{3.226}$$

One finds

$$\frac{1}{C_1\sqrt{5}} = \operatorname{sn}(C_1 + C_2, -1) \tag{3.227}$$

(i.e. $t_{\text{initial}} = 0$).

Let

$$\varPhi(t_{\text{end}}) = \frac{1}{\alpha_s}\left(\frac{1+\sqrt{5}}{2}\right) \tag{3.228}$$

(a true minimum—a “true” vacuum). One gets

$$\frac{1}{2} = -C_1 e^{H_0 t_{\text{end}}} \operatorname{sn}\left(C_1 e^{H_0 t_{\text{end}}} + C_2, -1\right). \tag{3.229}$$

Let us calculate the derivative of $\Phi(t)$. One gets

$$\frac{d\Phi}{dt} = -\frac{\sqrt{5}}{2\alpha_s} C_1 H_0 e^{H_0 t} \Big(\operatorname{sn}\left(C_1 e^{H_0 t} + C_2, -1\right) \\ + \operatorname{cn}\left(C_1 e^{H_0 t} + C_2, -1\right) \cdot \operatorname{dn}\left(C_1 e^{H_0 t} + C_2, -1\right) \Big). \tag{3.230}$$

Let us suppose a slow movement of Φ such that

$$\frac{d\Phi}{dt}(0) = 0. \tag{3.231}$$

From (3.231) one gets

$$\operatorname{sc}(C_1 + C_2, -1) = -\operatorname{dn}(C_1 + C_2, -1). \tag{3.232}$$

This gives us an equation for a sum of integration constants. Thus we can get from Eq. (3.232) and Eq. (3.227) integration constants C_1 and C_2 and from Eq. (3.229) t_{end}, which can give us an amount of inflation

$$\overline{N}_0 = H_0 t_{\mathrm{end}}. \tag{3.233}$$

In some sense Eq. (3.229) is an equation for an amount of inflation

$$\frac{1}{2} = -C_1 e^{\bar{N}_0} \operatorname{sn}\left(C_1 e^{\bar{N}_0} + C_2, -1\right). \tag{3.234}$$

Using Eq. (3.232) and Eq. (3.227) one easily gets

$$C_1 = -\sqrt{\frac{5-\sqrt{5}}{10}} \simeq -0.5256. \tag{3.235}$$

One gets

$$C_1 + C_2 = \frac{1}{\sqrt{2}} \left(F(90^\circ \backslash 45^\circ) - F(32^\circ \backslash 45^\circ)\right) = 0.9431 \tag{3.236}$$

where F is an elliptic integral of the first kind and

$$\arcsin\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right) \simeq \arcsin(0.5256) \simeq 32^\circ. \tag{3.237}$$

Let us come back to Eq. (3.234) and let us denote

$$e^{\bar{N}_0} = x_0. \tag{3.238}$$

One gets using (3.236) and (3.235)

$$C_2 \simeq 1.4687 \tag{3.239}$$

and finally

$$1.4866x_0 - 5.0354 = F\left(\varphi/45^\circ\right), \tag{3.240}$$

where

$$\cos\varphi = \frac{1.9026}{x_0}. \tag{3.241}$$

From Eq. (3.240) one finds

$$\frac{2.8284}{\cos\varphi} - 5.0354 = \int\limits_0^\varphi \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}. \tag{3.242}$$

We come back to this equation later.

Thus we get the following results. The Higgs field evolves from a “false” vacuum value ($\varPhi = 0$) to the “true” vacuum value ($\varPhi = \varPhi_0$) completing a second order phase transition. The potential of selfinteracting $\varPsi$ field changes and a new equilibrium $\varPsi_0$ is attained. However, the field $\varPsi$ evolves from $\varPsi_0$ to new value a little different from $\varPsi_0$ and afterwards a radiation era starts. The change of the value of a scalar field $\varPsi$ is small in terms of a variable y (see Eq. (2.69) for a definition) only from $\sqrt{\frac{n}{n+2}}$ to 1 (n is a natural number $n = \dim H$, and H is at least $G2$, $\dim G2 = 14$). Let us consider an evolution of a field $\varPsi$ in this epoch. From Eq. (2.59) we get

$$\ddot{\varphi} + \omega_0^2\varphi = 0 \tag{3.243}$$

where

$$\omega_0^2 = \frac{M_{\rm pl}^2}{2\overline{M}}(n+2)\left(\frac{n}{n+2}\right)^{\frac{n}{2}}|\gamma|\left(\frac{|\gamma|}{\beta}\right)^{\frac{n}{2}} \tag{3.244}$$

$$\varPsi = \varPsi_0 + \varphi \tag{3.245}$$

and we linearize Eq. (2.59) around $\varPsi_0$ neglecting a term with a first derivative of φ. Thus a scalar field $\varPsi$ undergoes small oscillations around the equilibrium $\varPsi_0$. These oscillations cannot be too long for a field should change from

$$\varPsi_0 = \ln\left(\sqrt{\frac{n|\gamma|}{(n+2)\beta}}\right) \tag{3.246}$$

to

$$\overline{\varPsi} = \ln\left(\sqrt{\frac{|\gamma|}{\beta}}\right). \tag{3.247}$$

In terms of a field φ from $\varphi = 0$ to $\varphi = \frac{1}{2}\ln(1+\frac{2}{n}) \simeq \frac{1}{n}$. Thus we have

$$\varphi = \varphi_0 \sin(\omega_0 t). \tag{3.248}$$

For $t = 0$, $\varphi = 0$ the time Δt to go from $\varphi = 0$ to $\varphi = \frac{1}{n}$ is simply equal to

$$\Delta t = \frac{\left|\arcsin(\frac{1}{\varphi_0 n})\right|}{\omega_0} \leq \frac{\pi}{2\omega_0}\,. \tag{3.249}$$

Thus the full amount of an inflation in this epoch is equal to

$$N_1 = \Delta t H_1 = \frac{H_1}{\omega_0}\left|\arcsin\left(\frac{1}{\varphi_0 n}\right)\right| \leq \frac{\pi H_1}{2\omega_0} = \frac{\pi M_{\rm pl}}{2\sqrt{6\overline{M}}(n+2)}\,. \tag{3.250}$$

In our notation

$$t_1 = t_r + \Delta t \tag{3.251}$$

(see Eq. (2.89)).

After a time t_1 the Universe goes to the phase of radiation domination with a strong interaction between a radiation and a scalar field Ψ up to a minimal temperature where an ordinary (a dust) matter appears. This matter evolves afterwards independently of a radiation and of a scalar field Ψ. The scalar field now is evolving as a quintessence.

In order to get some comparison let us consider an evolution of a scalar field Ψ during the second de Sitter phase in a slow-roll approximation. Using Eq. (2.59) one gets

$$\frac{dy}{dt} = \frac{\overline{B}y^{\frac{(n-2)}{2}}(y^2 - (\frac{n}{n+2}))}{\sqrt{1-y^2}} \tag{3.252}$$

or

$$\int \frac{\sqrt{1-y^2}}{y^{(\frac{n-2}{2})}(y^2 - (\frac{n}{n+2}))}\,dy = \overline{B}(t - t_0) \tag{3.253}$$

where

$$\overline{B} = 2(n+2)|\gamma| \left(\frac{n}{n+2}\right)^{\frac{n}{2}} \left(\frac{|\gamma|}{\beta}\right)^{\frac{n}{2}} \tag{3.254}$$

and

$$y = \sqrt{\frac{\beta}{|\gamma|}} \cdot e^{\Psi}. \tag{3.255}$$

Let

$$I = \int \frac{\sqrt{1-y^2}\,dy}{y^{(\frac{n-2}{2})}(y^2 - (\frac{n}{n+2}))} \simeq \int \frac{\sqrt{1-y^2}\,dy}{(y^2 - (\frac{n}{n+2}))} \tag{3.256}$$

for

$$\sqrt{\frac{n}{n+2}} \le y \le 1 \tag{3.257}$$

$$n > \dim G2 = 14. \tag{3.258}$$

For the integral I one finds:

$$\begin{aligned}
I = & -\arcsin(y) + \frac{1}{\sqrt{n(n+2)}}\sqrt{1-y^2} \\
& + \left(\sqrt{\frac{2}{n}} + \sqrt{\frac{n}{2}}\right)\frac{1}{n+2}\cdot\ln\left[\frac{2\left[\left(1+y\sqrt{\frac{n}{n+2}}\right)\left(y+\sqrt{\frac{n}{n+2}}\right)+\sqrt{\frac{2}{n+2}}\sqrt{1-y^2}\right]}{\left(y+\sqrt{\frac{n}{n+2}}\right)}\right] \\
& - \frac{1}{n+2}\left(\sqrt{\frac{2}{n}} + \sqrt{\frac{n}{2}}\right)\cdot\ln\left[\frac{2\left[\left(1-y\sqrt{\frac{n}{n+2}}\right)\left(y-\sqrt{\frac{n}{n+2}}\right)+\sqrt{\frac{2}{n+2}}\sqrt{1-y^2}\right]}{\left(y-\sqrt{\frac{n}{n+2}}\right)}\right].
\end{aligned} \tag{3.259}$$

In order to find an amount of an inflation let us find limits of I for $y = 1$ and $y = \sqrt{\frac{n}{n+2}}$. One finds

$$\begin{aligned}
\lim_{y\to 1} I = & -\frac{\pi}{2} + \left(\sqrt{\frac{2}{n}} + \sqrt{\frac{n}{2}}\right)\frac{\ln\left(2\left(1+\sqrt{\frac{n}{n+2}}\right)\right)}{(n+2)} \\
& - \left(\sqrt{\frac{2}{n}} + \sqrt{\frac{n}{2}}\right)\frac{\ln\left(2\left(1-\sqrt{\frac{n}{n+2}}\right)\right)}{(n+2)}
\end{aligned} \tag{3.260}$$

$$\lim_{y\to\sqrt{\frac{n}{n+2}}} I = -\infty. \tag{3.261}$$

Thus we see that a slow-roll approximation offers an infinite in time evolution of a field y from $\sqrt{\frac{n}{n+2}}$ to 1. It is similar to an evolution of a Higgs field in some slow-roll approximation schemes. The inflation never ends. Let us come back to the Eq. (2.110) and let us calculate a Hubble constant and a deceleration parameter for this model

$$H = \frac{\dot{R}}{R} = \left(\frac{2|\gamma|^{n+4}}{3\overline{M}\beta^n(n-1)}\right)\frac{(t-t_1)^3}{\left(1+\frac{|\gamma|^{n+4}}{3\overline{M}\beta^n(n-1)}(t-t_1)^4\right)} \tag{3.262}$$

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = -\frac{2}{3}\left(\frac{1}{(t-t_1)^4} + \frac{|\gamma|^{n+4}}{9\overline{M}\beta^n}\right). \tag{3.263}$$

Let us compare Eq. (3.262) with the similar equation for radiation dominated Universe in General Relativity

$$H_{\rm GR} = \frac{2}{3(t-t_1)} \tag{3.264}$$

and let us calculate a speed up factor

$$\varDelta_1 = \frac{H}{H_{\rm GR}} = \frac{|\gamma|^{n+4}}{\overline{M}\beta^n(n-1)} \frac{(t-t_1)^4}{\left(1+\frac{|\gamma|^{n+4}}{3\overline{M}\beta^n(n-1)}(t-t_1)^4\right)} \tag{3.265}$$

and let us do the same for a model Eq. (3.58) (i.e. radiation dominated with a quintessence). Using Eq. (3.59) one gets

$$\varDelta_2 = \frac{H}{H_{\rm GR}} = \frac{3\sqrt{3}m_{\rm pl}\rho_Q(t-t_1)}{2B\,{\rm tgh}\left(\frac{2\sqrt{3}\rho_Q}{B}m_{\rm pl}(t-t_1)\right)}\,. \tag{3.266}$$

Let us notice that for a $(t-t_1) \sim 0$

$$\varDelta_1 \simeq 0 \tag{3.267}$$

and

$$\lim_{t\to\infty} \varDelta_1 = +\infty. \tag{3.268}$$

Thus at early stages model Eq. (2.110) is slower than that in General Relativity. For large time the behaviour depends on details of the theory. In the case of model Eq. (3.58) we have

$$\lim_{t\to t_1} \varDelta_2 = \tfrac{3}{4} \tag{3.269}$$

and

$$\lim_{t\to\infty} \varDelta_2 = +\infty. \tag{3.270}$$

Thus in the first case $\varDelta$ is monotonically going from 0 to $+\infty$ and in the second case from $\frac{3}{4}$ to infinity. According to modern ideas an expansion rate in early Universe has an important influence on a production of light elements, the so called primordial abundance of light elements (see Ref. [11]). If at a beginning of primordial nucleosynthesis the Universe expansion rate is slower than in GR, then we have ^{4}He underproduction. This can be balanced by considering a larger ratio of a number density of barions to number density of photons (remember the barion number is a conserved quantity)

$$\eta = \frac{n_B}{n_\gamma} \tag{3.271}$$

where n_B, n_γ—barion and photon density numbers for a high redshift $z = 10^{10}$. In contrast to the latest if an expansion rate became faster than in GR during nucleosynthesis process, those bigger η (traditionally one uses the so called) η_{10}

$$\eta_{10} = 10^{10}\eta \tag{3.272}$$

do not result in excessive burning of deuterium because this happens in a shorter time. The standard model of big-bang nucleosynthesis (SBBN) demands

$$3 \le \eta_{10} \le 5.6. \tag{3.273}$$

It seems in the light of observational data from cosmic microwave background (CMB) (BOOMERANG and MAXIMA) and from the Lyman α-forest that $\eta_{10} = 8.8 \pm 1.4$, significantly higher than the SBBM value (for CMB) and 8.2 ± 2 or 12.3 ± 2 for Lyman α-forest. Thus the second model could help in principle to explain the data without modification of nuclear reaction rates due to neutrino degeneracy or introducing new decaying particles. As the Universe expands, cools and becomes more dilute, the nuclear reactions cease to create and destroy nuclei. The abundances of the light nuclei formed during this epoch are determined by the competition between a time available (an expansion rate) and a density of reactans: neutrons and protons.

The abundances of D, ^{3}He, ^{7}Li are limited by an expansion rate and are determined by the competition between the nuclear production/destruction rates and an (universal) expansion of the Universe. The SBBN theory is based on a flat radiation dominated Universe model in General Relativity and found in a laboratory nuclear reaction rates. Thus it is strongly constrained. Any significant discrepancy between observed and calculated value of η_{10} (known as baryometry) could be very dangerous. Thus changing the ratio of an universal expansion relative to the model of General Relativity can give some margin in a primordial alchemy.

In our theory after two inflationary phases we have in principle two radiation dominated phases described by Eq. (2.110) and Eq. (3.58) with speed up factors (3.265) and (3.266). However, the important point is to match those models. We get

$$\begin{aligned} R(t_d) &= R_0 \exp\left(H_0 t_r + H_1(t_1 - t_r)\right) \left(1 + \frac{\overline{r}}{(n-1)} \eta_{\min}^2\right)^{\frac{1}{2}} \\ &= \sqrt[4]{\frac{B}{\rho_Q}} \left(\operatorname{sh}\left(\frac{2\sqrt{3}\sqrt{\rho_Q}}{B} m_{\rm pl} \left(t_1 - \overline{t}_0 + \frac{\sqrt{2\overline{M}}\beta^{\frac{n}{4}}}{|\gamma|^{\frac{n+4}{4}}} \eta_{\min}^{\frac{1}{2}}\right)\right)\right)^{\frac{1}{2}} \end{aligned} \tag{3.274}$$

and

$$\begin{aligned} B = \rho_r(t_d) R^3(t_d) &= \rho_r \left(t_1 + \frac{\sqrt{2\overline{M}}\beta^{\frac{n}{4}}}{|\gamma|^{\frac{n+4}{4}}} \eta_{\min}^{\frac{1}{2}}\right) R^3 \left(t_1 + \frac{\sqrt{2\overline{M}}\beta^{\frac{n}{4}}}{|\gamma|^{\frac{n+4}{4}}} \eta_{\min}^{\frac{1}{2}}\right) \\ &= \rho_{\min} R_0^3 \exp\left(3H_0 t_r + 3H_1(t_1 - t_r)\right) \left(1 + \frac{\overline{r}}{(n+1)} \eta_{\min}^2\right)^{\frac{3}{2}} \end{aligned} \tag{3.275}$$

where $\rho_{\min}$ is given by

$$\rho_{\min} = M_{\rm pl}^2 \frac{\beta^{n+1}}{|\gamma|^n} \left(1 + \eta_{\min}\right)^{2(n+1)} W_4(1 + \eta_{\min}), \tag{3.276}$$

$W_4(y)$ is given by formula (2.90) and $\eta_{\rm min}$ by formula (2.97).

The time $t_r = t_{\rm end}$ in any inflation model for an evolution of the Higgs field in the first de Sitter phase. The time $t_1 = t_r + \varDelta t$ can be calculated from Eq. (3.249)

$$\varDelta t = \frac{\arcsin(\frac{1}{\varphi_0 n})}{\omega_0} \tag{3.277}$$

and is bounded by

$$\varDelta t \le \frac{1}{\omega_0} = \frac{1}{M_{\rm pl}|\gamma|^{\frac12}} \sqrt{\frac{2\overline{M}}{(n+2)}} \left(\frac{n+2}{n}\right)^{\frac n4} \left(\frac{\beta}{|\gamma|}\right)^{\frac n4}. \tag{3.278}$$

From Eq. (3.274) we can get the constant $\overline{t}_0$

$$\overline{t}_0 = t_1 + \frac{\sqrt{2\overline{M}}\beta^{\frac n4}}{|\gamma|^{\frac{n+4}{4}}} \eta_{\rm min}^{\frac12} - T \tag{3.279}$$

where

$$T = \frac{B\,\mathrm{arsh}\left[R_0^2\sqrt{\frac{\rho_Q}{B}}\exp\left(2H_0t_r + 2H_1(t_1-t_r)\right)\left(1+\frac{\overline{r}}{(n-1)}\eta_{\rm min}^2\right)\right]}{2\sqrt3\sqrt{\rho_Q}M_{\rm pl}} \tag{3.280}$$

and B is given by Eq. (3.275).

Let us notice that for $t = t_d$ the Universe undergoes a phase transition due to a change in an equation of state for a scalar field $\varPsi$ from $p_\varPsi = \frac43\rho_\varPsi$ in a first radiation dominated phase to $p_\varPsi = -\rho_\varPsi$ in a second radiation dominated phase (a quintessence phase).

A matter which appears in the second phase does not play any rôle in an evolution of the Universe. For $t = t_d$ we have a discontinuity in Hubble constants (parameters) for both phases

$$H(t_d) = \frac{2|\gamma|^{n+4}}{3\overline{M}(n-1)\beta^n} \cdot \frac{(t_d-t_1)^{\frac32}}{\left(1+\frac{\overline{r}}{(n-1)}\eta_{\rm min}^2\right)} \tag{3.281}$$

in the first phase and

$$\overline{H}(t_d) = \sqrt3 m_{\rm pl}\left(\frac{\sqrt{\rho_Q}}{B}\right)\mathrm{ctgh}\left(\frac{2\sqrt3}{B}\sqrt{\rho_Q}m_{\rm pl}T\right) \tag{3.282}$$

for the second one,

$$H(t_d) \ne \overline{H}(t_d) \tag{3.283}$$

$$\rho_Q = \frac12\lambda_{c0}(\varPsi_0) = \frac{|\gamma|^{\frac n2+1}n^{\frac n2}}{\beta^{\frac n2}(n+2)^{\frac n2+1}}. \tag{3.284}$$

Let us notice that a speed up factor $\mathit{\Delta}_1$ changes from

$$\mathit{\Delta}_1(t_1) = 0 \tag{3.285}$$

to

$$\mathit{\Delta}_1(t_d) = \frac{|\gamma|^{n+4}}{\overline{\overline{M}}(n-1)\beta^n} \cdot \frac{(t_d - t_1)^4}{\left(1 + \frac{\overline{r}}{(n-1)}\eta_{\min}^2\right)} \tag{3.286}$$

and an expansion rate can be slower than in General Relativity.

The speed up factor $\mathit{\Delta}_2$ has a more complicated behaviour than $\mathit{\Delta}_1$. It is natural to expect a rapid speed up of a rate of expansion resulting in non-equilibrium nuclear synthesis of light elements. Thus the presented scenario offers interesting possibilities for an evolution of early Universe. Recently some papers have appeared on scalar-tensor theories of gravitation exploiting the idea of a speed up factor in primordial nucleosynthesis [12], [13].

Finally let us come back to the model (3.64), (3.67). We cannot invert analytically the formula (3.67). Moreover taking under consideration Eqs (3.85–88) and making some natural simplifications we come to the following approximative formulae for $R(t)$.

For $x < 1.1969$ (Eq. (3.86))

$$R(t) = \sqrt[3]{\frac{A}{\rho_Q}}\,(0.44 + 0.085N) \tag{3.287}$$

where

$$N = \exp\left(0.374\frac{\sqrt{\rho_Q}}{M_{\rm pl}}(t - t_0) + 157.93\right). \tag{3.288}$$

One can calculate a Hubble parameter and a deceleration parameter and gets:

$$H = \frac{\dot{R}}{R} = \frac{\sqrt{\rho_Q}}{M_{\rm pl}}\frac{3.17N}{44 + 8.5N} \tag{3.289}$$

$$-q = \frac{\ddot{R}R}{\dot{R}^2} = \frac{5.21}{N} + 1. \tag{3.290}$$

We have $H > 0$, and $q < 0$. Thus the model expands and accelerates. First of all we consider

$$\rho_m = \frac{A}{R^3}e^{aQ}$$

and we take boundary for R. In this way one gets

$$\rho_m = a_i \cdot \rho_Q e^{aQ}, \quad i = 1, 2, 3, 4, \tag{3.291}$$

$$\begin{aligned} a_1 &= 2.015 \\ a_2 &= 0.034 \\ a_3 &= 1060 \\ a_4 &= 8 \end{aligned} \tag{3.292}$$

If we reconsider a contemporary gravitational constant G_N as $G_N e^{-(n+2)\Psi_0}$ we would get interesting ratios. Thus we get

$$\frac{\rho_m}{\rho_Q} = 0.034 \div 2.015 \quad \text{or} \quad 8 \div 1060 \tag{3.293}$$

which for the first is an excellent agreement with recent data concerning an acceleration of the Universe. Solving Eq. (3.288) for N if $R = 1.33\sqrt{\frac{A}{\rho_Q}}$ gives us

$$N = 10.47 \tag{3.294}$$

$$H = \frac{\sqrt{\rho_Q}}{M_{\text{pl}}} \cdot 0.25 \tag{3.295}$$

$$-q = 1.5. \tag{3.296}$$

Using Eq. (3.288) and Eq. (3.295) we can estimate a time of our contemporary epoch

$$t = -\frac{1}{h} \cdot 10^{15}\text{yr} - t_0 \tag{3.297}$$

where h is a dimensionless Hubble parameter, $H = h \cdot H_0$ (we take for a contemporary Hubble parameter $H_0 = 100\frac{\text{km/s}}{\text{Mps}}$), $0.7 < h < 1$, which seems to be too much.

We can also estimate a density of a quintessence

$$\rho_Q = \frac{H^2}{G_N} \cdot 0.571 = h^2 \cdot 0.08 \cdot 10^{-23}\frac{\text{kg}}{\text{m}^3} = 8h^2 \cdot 10^{-29}\frac{\text{g}}{\text{cm}^3}\,. \tag{3.298}$$

Let us come back to the Eqs (3.177) and (3.184). In Eq. (3.184) a Higgs field is given by a P Weierstrass function. This function can be expressed by Jacobi elliptic function

$$P(z) = e_3 + (e_1 - e_2)\,\text{ns}^2\left(z(e_1 - e_2)^{\frac{1}{2}}\right) \tag{3.299}$$

with

$$K^2 = \frac{e_2 - e_3}{e_1 - e_3} \tag{3.300}$$

where e_1, e_2, e_3 are roots of the polynomial

$$4x^3 - g_2 x - g_3. \tag{3.301}$$

Thus we should solve a cubic equation

$$x^3 - \frac{1}{12}\left(5 - 3a^2\right) - \frac{1}{12}\left(\frac{a^2}{4} + \frac{10}{9}\right) = 0. \tag{3.302}$$

We want Eq. (3.302) to have all real roots. Thus we need

$$p = -\frac{1}{12}\left(5 - 3a^2\right) < 0 \tag{3.303}$$

and

$$D = \frac{p^3}{27} + \frac{q^2}{4} < 0 \tag{3.304}$$

where

$$q = -\frac{1}{12}\left(\frac{a^2}{4} + \frac{10}{9}\right). \tag{3.305}$$

Both conditions (3.303) and (3.304) are satisfied if

$$0 < a < 0.3115\,. \tag{3.306}$$

In this case we get

$$e_1 = \frac{1}{3}\sqrt{5-3a^2}\cos\frac{\varphi}{3} \tag{3.307}$$

$$e_2 = \frac{1}{3}\sqrt{5-3a^2}\cos\frac{\varphi+2\pi}{3} \tag{3.308}$$

$$e_3 = \frac{1}{3}\sqrt{5-3a^2}\cos\frac{\varphi+4\pi}{3} \tag{3.309}$$

where

$$\cos\varphi = -\frac{(9a^2+40)}{12(5-3a^2)^{3/2}}\,. \tag{3.310}$$

From (3.307–309) one gets

$$K^2 = \frac{\sin\frac{\varphi}{3}}{\sin\frac{\varphi+2\pi}{3}} \tag{3.311}$$

$$e_1 > e_2 > e_3 \tag{3.312}$$

Thus

$$\varPhi(t) = \frac{(5+3a)\sqrt{5+4a}}{8\alpha_s(A_1\,\mathrm{ns}^2(u)+A_2)} + \frac{\left(1+\sqrt{5+4a}\right)}{2\alpha_s} \tag{3.313}$$

where

$$u = \sqrt[4]{\frac{Ae^{n\varPsi_1}\left(5Ae^{n\varPsi_1}-3\alpha_s^2r^2\right)}{r^4}}\sqrt{\sin\frac{\varphi+\pi}{3}}(t-t_0) \tag{3.314}$$

$$A_1 = e_1 - e_2 = \sqrt{\frac{5}{3}-a^2}\,\sin\frac{\varphi+\pi}{3} \tag{3.315}$$

$$A_2 = e_3 + \frac{1}{12}(5+6a) = \sqrt{\frac{5}{3}-a^2}\,\cos\frac{\varphi+4\pi}{3} + \frac{1}{12}(5+6a) \tag{3.316}$$

and

$$\cos\varphi = -\frac{\sqrt{A}e^{\frac{n}{2}\Psi_1}\left(9\alpha_s^2 Br^2 + 40Ae^{n\Psi_1}\right)}{12\left(5Ae^{n\Psi_1} - 3\alpha_s^2 Br^2\right)^{3/2}} \tag{3.317}$$

$$0 < \sqrt{\frac{\alpha_s^2 Br^2}{Ae^{n\Psi_1}}} < 0.3115 \tag{3.318}$$

and a is given by Eq. (3.173).

The function $\mathrm{ns}(u, K)$ is an elliptic Jacobi function with a modulus K given by Eq. (3.311)

$$\mathrm{ns}(u, K) = \frac{1}{\mathrm{sn}(u, K)} \tag{3.319}$$

$$\mathrm{sn}(u, K) = \mathrm{sinam}(u, K). \tag{3.320}$$

In order to justify our treatment of Eqs (3.130) and (3.133) we consider the full field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2\overline{\Lambda}u_\mu u_\nu - \overline{\Lambda}g_{\mu\nu} \tag{3.321}$$

(where $\overline{\Lambda}$ is given by Eq. (3.136)).

However, in this case we consider $\overline{\Lambda}$ arbitrary. Let us consider static and spherically symmetric metric

$$ds^2 = e^v\,dt^2 - e^\lambda\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \tag{3.322}$$

where $v = v(r)$ and $\lambda = \lambda(r)$.

From Eqs (3.321–322) one gets

$$r^2\overline{\Lambda} = e^{-\lambda}(rv' + 1) - 1 \tag{3.323}$$

$$r^2\overline{\Lambda} = e^{-\lambda}(r\lambda' - 1) + 1 \tag{3.324}$$

$$\overline{\Lambda}' = -\overline{\Lambda}v' \tag{3.325}$$

where $'$ means derivation with respect to r. Summing up (3.323) and (3.324) one gets

$$e^{-\lambda}(\lambda' + v') = 2\overline{\Lambda}r. \tag{3.326}$$

Moreover in our case $\overline{\Lambda}$ is very small. Thus we get approximately

$$\frac{d\overline{\Lambda}}{dr} \simeq 0 \tag{3.327}$$

and

$$\lambda' \cong -v'. \tag{3.328}$$

In this way we go to the solution (3.137) for a small $\overline{\Lambda}$ which is our case. In this way $\lambda = -v$ and

$$B(r) = e^{v} = e^{-\lambda}. \tag{3.329}$$

Let us check a consistency of our solution. In order to do this we consider Eqs (3.323–325) in full. One gets from Eq. (3.325)

$$\overline{\Lambda} = e^{-\mu_0} e^{-v}. \tag{3.330}$$

For $\overline{\Lambda}$ is very small we should suppose that μ_0 is positive and very large ($\mu_0 \to \infty$). From Eqs (3.323–324) one gets

$$\frac{d}{dr}(\lambda + v) = 2e^{-\mu_0} \cdot r \cdot e^{(\lambda - v)}. \tag{3.331}$$

Supposing that

$$\lambda(r_0) + v(r_0) = 0 \tag{3.332}$$

for an established $r_0 > 0$ (r_0 is greater than a Schwarzschild radius of the mass M_0 from the solution (3.137)). Taking sufficiently big μ_0 we get approximately

$$\frac{d}{dr}(\lambda + v) \simeq 0 \tag{3.333}$$

and of course

$$\lambda \cong -v. \tag{3.334}$$

Let us come back to the Eq. (3.242) and consider it for

$$\varphi = k\pi + \tfrac{\pi}{2} - \varepsilon, \quad k = 0, \pm 1, \pm 2, \dots , \tag{3.335}$$

where ε is considered to be small. One gets

$$\frac{2.8284(-1)^k}{\sin\varepsilon} - 5.0354 = \int_0^{\frac{\pi}{2}-\varepsilon} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} + 2kK\left(\tfrac{1}{2}\right), \tag{3.336}$$

where

$$K\left(\tfrac{1}{2}\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} = 1.8541 \tag{3.337}$$

is a full elliptic integral of the first kind for a modulus equal $\frac{1}{2}$. For a small ε we can write

$$\sin\varepsilon \simeq \varepsilon \tag{3.338}$$

$$\int_0^{\frac{\pi}{2}-\varepsilon} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} \cong K\left(\tfrac{1}{2}\right) - \sqrt{2}\,\varepsilon. \tag{3.339}$$

Thus one gets for $k = 2l$

$$\frac{2.8284}{\varepsilon} - 5.0354 = -\sqrt{2}\,\varepsilon + (4l+1)1.8541. \tag{3.340}$$

It is easy to notice that we can realize a 60-fold inflation in our model for

$$x_0 = \frac{1.9026}{|\varepsilon|} \tag{3.341}$$

can be arbitrary big for sufficiently big l.

Let us take $l = 25 \cdot 10^{24}$. In this case we have for ε an equation

$$\varepsilon^2 - \big((4l+1)1.3110 + 5.0354\big)\varepsilon + 2 = 0 \tag{3.342}$$

or

$$\varepsilon^2 - \left(10^{26} + 6.0354\right)\varepsilon + 2 = 0, \tag{3.342a}$$

i.e.

$$\varepsilon^2 - 10^{26}\varepsilon + 2 = 0, \tag{3.342b}$$

and finally

$$|\varepsilon| \simeq 10^{-26} \tag{3.343}$$

$$x_0 \simeq 1.9026 \cdot 10^{26} \tag{3.344}$$

$$\ln x_0 \simeq 60.51 \tag{3.345}$$

which gives us a 60-fold inflation.

Moreover the Eq. (3.242) has an infinite number of solutions for $0 \le \psi \le \frac{\pi}{2}$, $\varphi = k\pi + \psi$,

$$\frac{2.8284}{\cos\psi} - 5.0354 = \int\limits_0^\psi \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} + 7.5164 \cdot l \tag{3.346}$$

with

$$x_0 = \frac{1.9026}{\cos\psi}, \quad l = 0, 1, 2, 3, \dots, \quad k = 2l. \tag{3.347}$$

One can find roots of Eq. (3.346) for
$l = 5$

$$\begin{aligned} x_0 &= 16.83 \\ \ln x_0 &= 2.82, \end{aligned} \tag{3.348a}$$

$l = 50$

$$\begin{aligned} x_0 &= 0.24 \cdot 10^3 \\ \ln x_0 &= 5.48 \end{aligned} \tag{3.348b}$$

and $\ln x_0$ for $l = 5 \cdot 10^{24}$. In the last case one finds using 70-digit arithmetics

$$\ln x_0 \simeq 58.479\ldots \tag{3.349}$$

which gives us an almost 60-fold inflation. In general an amount of inflation

$$\overline{N}_0 = \ln x_0 \tag{3.350}$$

is a function of $l = 0, 1, 2, \ldots$ and probably can be connected to the Dirac large number hypothesis.

Finally let us take $l = 10^n$ where $n > 10$. In this case one can solve Eq. (3.342a) and get

$$|\varepsilon| \simeq \frac{2}{[(4 \cdot 10^n + 1)1.3110 + 5.0354]}\,. \tag{3.351}$$

Thus for x_0 we find

$$x_0 \simeq 5 \cdot 10^n \tag{3.352}$$

and

$$\overline{N}_0(n) \simeq 1.60 + 2.30n. \tag{3.353}$$

In this way, for large l, $\overline{N}_0$ is a linear function of a logarithm of l,

$$\begin{aligned} &\overline{N}_0(10) \simeq 24.6,\ \overline{N}_0(20) \simeq 47.6,\ \overline{N}_0(24) \simeq 56.8,\\ &\overline{N}_0(25) \simeq 59.1,\ \overline{N}_0(26) \simeq 61.4 \end{aligned} \tag{3.354}$$

or

$$\overline{N}_0(\log_{10} l) = 1.6 + \ln l. \tag{3.355}$$

It is easy to see that Eq. (3.355) is an excellent approximation even for $l = 50$.

Let us come back to the Eq. (3.29) in order to find a power spectrum for our simplified model of inflation. Using Eqs (3.161), (3.230), (3.235) and (3.239) one gets after some algebra

$$P_R(K) \cong 2.8944 \left(\frac{H_0}{2\pi}\right)^2 n_1 \alpha_s^2 f\left(\frac{K}{R_0 H_0}\right) \tag{3.356}$$

where

$$\begin{aligned} f(x) = x^{-2}\Big(&\mathrm{sn}(1.4687 - 0.5256x, -1)\\ &+ \mathrm{cn}(1.4687 - 0.5256x, -1)\,\mathrm{dn}(1.4687 - 0.5256x, -1)\Big)^{-2}. \end{aligned} \tag{3.357}$$

Using some relations among elliptic functions one gets

$$f(x) = \frac{1}{x^2} \cdot \frac{2\,\mathrm{dn}^4(u, \frac12)}{\left(\mathrm{sn}(u, \frac12)\,\mathrm{dn}(u, \frac12) + \sqrt{2}\ \mathrm{cn}(u, \frac12)\right)^2} \tag{3.358}$$

where

$$u = 2.0770 - 0.7433x. \tag{3.359}$$

Let us consider a more general situation for the Eq. (3.231), i.e.

$$\frac{d\Phi}{dt}(0) = h \tag{3.360}$$

where $h \neq 0$. In this way one gets

$$h = -\frac{\sqrt{5}}{\alpha_s} C_1 H_0 \big(\mathrm{sn}(C_1 + C_2, -1) + \mathrm{cn}(C_1 + C_2, -1)\,\mathrm{dn}(C_1 + C_2, -1)\big). \tag{3.361}$$

Using Eq. (3.227) one gets

$$\begin{aligned} C_2 = {} & \frac{K(\frac{1}{2})}{\sqrt{2}} - \frac{\sqrt{10}}{\sqrt{\sqrt{\left(1+\frac{\alpha_s h}{H_0}\right)^4 + 4} - \left(1+\frac{\alpha_s h}{H_0}\right)^2}} \\ & - \frac{1}{\sqrt{2}} \int_0^{\arccos\sqrt{\frac{\sqrt{\left(1+\frac{\alpha_s h}{H_0}\right)^4+4} - \left(1+\frac{\alpha_s h}{H_0}\right)^2}{2}}} \frac{d\varphi}{\sqrt{1 - \frac{1}{2}\sin^2\varphi}} \end{aligned} \tag{3.362}$$

and

$$|C_1| = \frac{\sqrt{10}}{\sqrt{\sqrt{\left(1+\frac{\alpha_s h}{H_0}\right)^4 + 4} - \left(1+\frac{\alpha_s h}{H_0}\right)^2}}\,. \tag{3.363}$$

Let us come back to the Eq. (3.229) to find an amount of inflation for C_2 and C_1 given by Eqs (3.362–363). One gets

$$\sqrt{2}C_2 - K\left(\tfrac{1}{2}\right) = \int_0^{\arccos\left(\frac{1}{2C_1 e^{N_0}}\right)} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} - \sqrt{2}C_1 e^{N_0}, \tag{3.364}$$

or using a natural substitution

$$\cos\varphi = \frac{1}{2C_1 e^{N_0}}\,, \tag{3.365}$$

$$\sqrt{2}C_2 - K\left(\tfrac{1}{2}\right) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} - \frac{\sqrt{2}}{2\cos\varphi}\,. \tag{3.366}$$

Taking as usual

$$\varphi = 2l\pi + \tfrac{\pi}{2} - \varepsilon, \quad l = 0, 1, 2, \dots\,, \tag{3.367}$$

one gets for small ε

$$\varepsilon \simeq \frac{1}{\sqrt{2}(4l+2)K(\frac{1}{2}) - 2C_2} \simeq \frac{1}{(4l+2)\sqrt{2}K(\frac{1}{2})} \tag{3.368}$$

(for large l) and eventually

$$\begin{aligned} e^{N_0} &\simeq \frac{4\sqrt{2}K(\frac{1}{2})}{C_1} l \\ N_0 &= \ln\left(\frac{4\sqrt{2}K(\frac{1}{2})}{C_1}\right) + \ln l. \end{aligned} \tag{3.369}$$

Let us calculate $\frac{d\Phi}{dt}$ for $t = t^* = \frac{1}{H_0}\ln\bigl(\frac{K}{R_0 H_0}\bigr)$. One gets

$$\begin{aligned} \frac{d\Phi}{dt}(t^*) = -\frac{\sqrt{5}}{2\alpha_s} C_1 H_0 \left(\frac{K}{R_0 H_0}\right) \Bigl(\operatorname{sn}\Bigl(C_1\Bigl(\frac{K}{R_0H_0}\Bigr) + C_2, -1\Bigr) \\ + \operatorname{cn}\Bigl(C_1\Bigl(\frac{K}{R_0H_0}\Bigr) + C_2, -1\Bigr) \operatorname{dn}\Bigl(C_1\Bigl(\frac{K}{R_0H_0}\Bigr) + C_2, -1\Bigr)\Bigr). \end{aligned} \tag{3.370}$$

Thus we can write down a $P_R(K)$ function.

$$P_R(K) = \left(\frac{H_0}{2\pi}\right)^2 \cdot \frac{4\alpha_s^2 n_1}{5C_1^2} f\left(\frac{K}{R_0 H_0}\right) \tag{3.371}$$

where

$$f(x) = \frac{1}{x^2}\bigl(\operatorname{sn}(C_1x + C_2, -1) + \operatorname{cn}(C_1x + C_2, -1)\operatorname{dn}(C_1x + C_2, -1)\bigr)^{-2}. \tag{3.372}$$

Using some relation among elliptic functions one finds

$$P_R(K) = \left(\frac{H_0}{2\pi}\right)^2 \cdot \frac{4\alpha_s^2 n_1}{5C_1^2} g\left(\frac{K}{R_0 H_0}\right) \tag{3.373}$$

where

$$g(x) = \frac{1}{x^2} \cdot \frac{2}{\bigl(\operatorname{sd}(u, \frac{1}{2}) + \sqrt{2}\operatorname{cd}(u, \frac{1}{2})\operatorname{nd}(u, \frac{1}{2})\bigr)^2} \tag{3.374}$$

and

$$u = \sqrt{2}C_1 x + \sqrt{2}C_2\,. \tag{3.375}$$

Moreover we can reparametrize (3.374–375) in the following way

$$u = \sqrt{2}C_1 x + K(\tfrac{1}{2}) - C_1\sqrt{2} - \int_0^{\arccos\left(\frac{\sqrt{5}}{C_1}\right)} \frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}} \tag{3.376}$$

where

$$\frac{\alpha_s h}{H_0} = -1 \pm \frac{1}{|C_1|}\sqrt{\frac{C_1^4 - 25}{5}} \tag{3.377}$$

$$|C_1| > \sqrt{5}. \tag{3.378}$$

For large C_1 one gets

$$u \cong \sqrt{2}C_1(x-1) \tag{3.379}$$

$$\frac{\alpha_s h}{H_0} \simeq \frac{C_1}{\sqrt{5}}. \tag{3.380}$$

Using (3.369),

$$u = \frac{4\sqrt{2}K(\frac{1}{2})l}{e^{N_0}}\,(x-1). \tag{3.381}$$

If we take large C_1 (it means, a large h)

$$h = \frac{C_1 H_0}{\sqrt{5}\alpha_s} \tag{3.382}$$

and simultaneously sufficiently large l we can achieve a 60-fold inflation with a function $P_R(K)$ given by the formula (3.374). Large C_1 means here $C_1 \simeq 100$, large l means $l \simeq 10^{25}$.

Let us calculate the spectral index for our $P_R(K)$ function, i.e.

$$n_s(K) - 1 = \frac{d\ln P_R(K)}{d\ln K}. \tag{3.383}$$

One gets

$$n_s(K) - 1 = -2 - 2C_1 x\,\frac{\sqrt{2}\,\mathrm{cd}(u,\frac{1}{2}) - \mathrm{sd}^3(u,\frac{1}{2})}{\mathrm{sd}(u,\frac{1}{2}) + \sqrt{2}\,\mathrm{cd}(u,\frac{1}{2})\,\mathrm{nd}(u,\frac{1}{2})} \tag{3.384}$$

where

$$\begin{aligned} u &= \sqrt{2}(C_1 x + C_2) = \frac{\sqrt{2}}{R_0 H_0}\,(C_1 K + C_2 R_0 H_0) \\ x &= \frac{K}{R_0 H_0} \end{aligned} \tag{3.385}$$

A very interesting characteristic of $P_R(K)$ is also $\dfrac{dn_s(K)}{d\ln K}$. One gets

$$\begin{aligned}\frac{dn_s(K)}{d\ln K} &= -2C_1 x\,\frac{\sqrt{2}\,\mathrm{cd}(u,\frac12)-\mathrm{sd}^3(u,\frac12)}{\mathrm{sd}(u,\frac12)+\sqrt{2}\,\mathrm{cd}(u,\frac12)\,\mathrm{nd}(u,\frac12)}\\ &\quad - 2C_1^2x^2\,\mathrm{sd}^2(u,\tfrac12) - 2C_1^2x^2\frac{\left(\sqrt{2}\,\mathrm{cd}(u,\frac12)-\mathrm{sd}^3(u,\frac12)\right)^2}{\left(\mathrm{sd}(u,\frac12)+\sqrt{2}\,\mathrm{cd}(u,\frac12)\,\mathrm{nd}(u,\frac12)\right)^2}\end{aligned} \tag{3.386}$$

where u and x are given by Eq. (3.385).

Let us take for a trial $C_1 = -0.5256$ and $C_2 = 1.4687$. In this case one finds

$$n_s(K)-1 = -2 + 1.0512x\,\frac{\sqrt{2}\,\mathrm{cd}(u,\frac12)-\mathrm{sd}^3(u,\frac12)}{\mathrm{sd}(u,\frac12)+\sqrt{2}\,\mathrm{cd}(u,\frac12)\,\mathrm{nd}(u,\frac12)} \tag{3.387}$$

$$\begin{aligned}\frac{dn_s(K)}{d\ln K} &= 1.0512x\,\frac{\sqrt{2}\,\mathrm{cd}(u,\frac12)-\mathrm{sd}^3(u,\frac12)}{\mathrm{sd}(u,\frac12)+\sqrt{2}\,\mathrm{cd}(u,\frac12)\,\mathrm{nd}(u,\frac12)}\\ &\quad - 0.5525x^2\,\mathrm{sd}^2(u,\tfrac12) - 0.5525x^2\frac{\left(\sqrt{2}\,\mathrm{cd}(u,\frac12)-\mathrm{sd}^3(u,\frac12)\right)^2}{\left(\mathrm{sd}(u,\frac12)+\sqrt{2}\,\mathrm{cd}(u,\frac12)\,\mathrm{nd}(u,\frac12)\right)^2}\end{aligned} \tag{3.388}$$

$$\begin{aligned} u &= 2.0771 - 0.7433x\\ x &= \frac{K}{R_0H_0}\end{aligned} \tag{3.388a}$$

The interesting point is to find $n_s(K) \simeq 1$ (a flat power spectrum). One gets

$$\mathrm{sd}(u,\tfrac12)+\sqrt{2}\,\mathrm{cd}(u,\tfrac12)\,\mathrm{nd}(u,\tfrac12) = C_1x\left(\mathrm{sd}^3(u,\tfrac12)-\sqrt{2}\,\mathrm{cd}(u,\tfrac12)\right) \tag{3.389}$$

and

$$\mathrm{sd}(u,\tfrac12)+\sqrt{2}\,\mathrm{cd}(u,\tfrac12)\,\mathrm{nd}(u,\tfrac12) \neq 0. \tag{3.390}$$

Using (3.389–390) one gets

$$\frac{dn_s}{d\ln K} = -2C_1^2x^2\,\mathrm{sd}^2(u,\tfrac12) \tag{3.391}$$

if Eqs (3.389–390) are satisfied.

In the case of special C_1 and C_2 one gets

$$\frac{dn_s}{d\ln K} = -0.5525x^2\,\mathrm{sd}^2(u,\tfrac12) \tag{3.392}$$

where x, u are given by Eq. (3.388a).

Let us reparametrize the Eq. (3.389). One gets

$$\operatorname{sd}(u,\tfrac{1}{2})+\sqrt{2}\operatorname{cd}(u,\tfrac{1}{2})\operatorname{nd}(u,\tfrac{1}{2})=\frac{u-\sqrt{2}C_2}{\sqrt{2}}\left(\operatorname{sd}^3(u,\tfrac{1}{2})-\sqrt{2}\operatorname{cd}(u,\tfrac{1}{2})\right). \tag{3.393}$$

For sufficiently big u one gets (the equation has infinite number of roots)

$$u=u_1+2lK(\tfrac{1}{2}),\qquad l=0,\pm1,\pm2,\ldots \tag{3.394}$$

where u_1 satisfies the equation

$$\operatorname{sn}(u_1,\tfrac{1}{2})=\frac{1}{3}\left(1+\sqrt[3]{5}\left(\sqrt[3]{65+\sqrt{20357}}-\sqrt[3]{\sqrt{20357}-65}\right)\right)^{\frac{1}{2}}\simeq 0.65\ldots \tag{3.395}$$

Moreover for $x=\frac{K}{R_0H_0}>0$ we have the condition

$$\frac{u_1+2lK(\frac{1}{2})-\sqrt{2}C_2}{\sqrt{2}C_1}>0. \tag{3.396}$$

One finds

$$\operatorname{sd}^2(u_1,\tfrac{1}{2})=0.53\ldots \tag{3.397}$$

Thus

$$\frac{dn_s}{d\ln K}\left(u_1+2lK(\tfrac{1}{2})\right)\simeq -0.53\left(\frac{u_1+2lK(\frac{1}{2})-\sqrt{2}C_2}{\sqrt{2}C_1}\right)^2, \qquad l=0,\pm1,\pm2,\ldots \tag{3.398}$$

One gets

$$\arcsin 0.65\simeq 40^\circ.54 \tag{3.399}$$

$$u_1\cong\int_0^{40^\circ.54}\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}=F(40^\circ.54/45^\circ)\cong 0.73\ldots \tag{3.400}$$

Taking special value of C_1 and C_2 one gets

$$x\cong 1.81-4.68l \tag{3.401}$$

where $l=0,\pm1,\pm2,\ldots$. For $x>0$ it is easy to see that l should be nonpositive. Practically l should be a negative integer

$$l<-3 \tag{3.402}$$

and

$$\frac{dn_s}{d\ln K} \cong -0.29(1.81 - 4.68l)^2. \tag{3.403}$$

Thus for

$$K_l = R_0 H_0 (1.81 - 4.68l)$$

we get

$$n_s(K_l) \cong 1. \tag{3.404}$$

The important range of $\ln K$, $\Delta \ln K$ is about 10.

Thus

$$n_s(K) \simeq 1 \pm 2.9(1.81 - 4.68l)^2. \tag{3.405}$$

Taking $l = -3$ one gets

$$-300 < n_s(K) < 300 \tag{3.406}$$

and

$$P_R(K) \sim K^{1-n_s(K)}$$

is not flat in the range considered.

Moreover we can improve the results considering Eq. (3.398) for large C_1. For large C_1 one gets

$$\frac{C_2}{C_1} \simeq -1. \tag{3.407}$$

Thus we find

$$\frac{dn_s}{d\ln K} \simeq -1.06 \left(u_1 + 2lK(\tfrac{1}{2}) + C_1\right)^2. \tag{3.408}$$

Taking large value of C_1 in such a way that

$$C_1 = -2lK(\tfrac{1}{2}) - u_1 + \varepsilon \tag{3.409}$$

where ε is a small number,

$$\varepsilon \simeq 10^{-n},$$

one gets

$$\frac{dn_s}{d\ln K} \simeq -1.06 \cdot 10^{-2n}. \tag{3.410}$$

The last condition means that we should take

$$h \cong \frac{H_0|C_1|}{\sqrt{5}\alpha_s} = \frac{H_0}{\sqrt{5}\alpha_s} \left(0.73 - 2lK(\tfrac{1}{2}) - 10^{-n}\right) \tag{3.411}$$

where l is an integer, $l \simeq -100$.

Thus for some special values of h we can get arbitrarily small $\frac{dn_s}{d\ln K}$ which means we can achieve a flat power spectrum in the range considered ($\Delta \ln K \sim 10$),

$$n_s(K) = 1 \pm 10^{-(2n-1)}, \tag{3.412}$$

n arbitrarily big.

Let us consider the value of K coresponding to our value of $n_s(K)=1$. One gets

$$x=\frac{0.73+2lK(\frac{1}{2})-\sqrt{2}C_2}{\sqrt{2}C_1}\,. \tag{3.413}$$

Using our assumption on a large C_1 and Eq. (3.409) one finds

$$x\cong\left(1-\frac{1}{\sqrt{2}}\right)-\frac{\varepsilon}{2lK(\frac{1}{2})} \tag{3.414}$$

or (for ε is small and l quite big)

$$x\simeq 0.293 \tag{3.415}$$

and

$$K\simeq 0.3R_0H_0 \tag{3.416}$$

with the range $\Delta\ln K\sim 10$. It means that

$$0.3e^{-10}\le\frac{K}{R_0H_0}\le 0.3\cdot e^{10} \tag{3.417}$$

which gives us a full cosmologically interesting region

$$10^{-4}\le\frac{K}{R_0H_0}\le 10^4. \tag{3.418}$$

Finally let us consider in this section two problems. First we match our solutions of an evolution of the Universe, i.e. both de Sitter phases, radiation dominated Universe, matter dominated Universe and K-essence Universe (kinetic energy dominated quintessence). In order to simplify calculations we match the first de Sitter phase to radiation dominated Universe using some issues from the second de Sitter phase. Secondly we write down solution to the evolution of Universe with a cosmological constant (from the second de Sitter phase), with a radiation and with a matter (a dust). All of these material ingredients are evolving independently in this case, similarly as a matter (dust) and a quintessence for the model (3.67).

Let us consider Eq. (2.26) and let us match it to Eq. (3.58). One gets

$$R_1=\sqrt[4]{\frac{B}{\rho_Q}}\sqrt{\mathrm{sh}\left(\frac{2\sqrt{3}\sqrt{\rho_Q}M_{\mathrm{pl}}}{B}t_{\mathrm{end}}\right)} \tag{3.419}$$

$$R_1=R_0e^{H_0t_{\mathrm{end}}} \tag{3.420}$$

where

$$t_{\mathrm{end}}=\frac{N_0}{H_0} \tag{3.421}$$

is the time of the end of the first de Sitter phase. (N_0 is the amount of an inflation.)

Simultaneously we need a local conservation of an energy, i.e.

$$\widetilde{\rho}_r(R_1) = \frac{B}{R_1^4} \tag{3.422}$$

where

$$\widetilde{\rho}_r(R_1) = 6\left(H_0^2 - H_1^2\right); \tag{3.423}$$

see the formula

$$\rho_r = \lambda_{c1}(\Psi_0) - \lambda_{c0}(\Psi_1) = \lambda_{c0}(\Psi_1)\left(\frac{H_0^2}{H_1^2} - 1\right).$$

After some calculations one gets

$$B = \frac{2\sqrt{3}M_{\rm pl}\sqrt{\rho_Q}t_{\rm end}}{\operatorname{arsh}\left(\sqrt{\frac{\rho_Q}{\widetilde{\rho}_r}}\right)}. \tag{3.424}$$

Now we need match a solution with a radiation dominated period to the solution with matter dominated period (with the same cosmological constant—quintessence energy density). One gets

$$\frac{A}{R_2^3} + \rho_Q = \frac{B}{R_2^4} + \rho_Q \tag{3.425}$$

and finally

$$A = \frac{B}{R_2} \tag{3.426}$$

where

$$R_2 = 0.7916\sqrt[3]{\frac{A}{\rho_Q}} = \sqrt[4]{\frac{B}{\rho_0}}\sqrt{\operatorname{sh}\left(\frac{2\sqrt{3}\sqrt{\rho_Q}M_{\rm pl}}{B}\,t_1\right)}, \tag{3.427}$$

t_1 is a time to match (see (3.90)). Let us consider

$$N_1 = H_1\left(t_1 - t_{\rm end}\right) \tag{3.428}$$

as an amount of an inflation during the second de Sitter phase. In this way one gets

$$t_1 = \frac{N_1}{H_1} + \frac{N_0}{H_0} \tag{3.429}$$

and finally

$$A = 2.015\cdot\frac{M_{\rm pl}^{3/4}\rho_Q^{5/8}\left(\frac{N_0}{H_0}\right)^{3/4}}{\operatorname{arsh}^{4/3}\sqrt{\frac{\rho_Q}{\widetilde{\rho}_r}}}\operatorname{sh}^{3/2}\left(\sqrt{\frac{\rho_Q}{\widetilde{\rho}_r}}\left(\frac{N_1}{N_0}\frac{H_0}{H_1} + 1\right)\right) \tag{3.430}$$

$$B = \frac{2\sqrt{3}\,M_{\rm pl}\sqrt{\rho_Q}}{\operatorname{arsh}\left(\sqrt{\frac{\rho_Q}{\widetilde{\rho}_r}}\right)}\,\frac{N_0}{H_0}. \tag{3.431}$$

However, we should match also a time. In this way we get from Eqs (3.287–288)

$$t_0 = \frac{418.48}{\sqrt{\rho_Q}} M_{\rm pl} + \frac{N_1}{H_1} \tag{3.432}$$

or using the value $\rho_Q = \lambda_{c0} = 10^{-52} \frac{1}{\rm m^2}$ we get

$$t_0 = 0.447 \cdot 10^{14}{\rm yr} + \frac{N_1}{H_1}\,. \tag{3.433}$$

If R_0 is an initial value of "a radius" of the Universe we get

$$\begin{aligned} R_0 = R_1 e^{-N_0} &= \sqrt[4]{\frac{2\sqrt{3}\,M_{\rm pl}\sqrt{\rho_Q}}{\widetilde{\rho}_r \,{\rm arsh}\left(\sqrt{\frac{\rho_Q}{\widetilde{\rho}_r}}\right)}} \cdot \exp\left(-\left(N_0 + \frac{n+2}{4}\Psi_0\right)\right) \\ &= \sqrt[4]{\frac{2\sqrt{3}\,M_{\rm pl}\sqrt{\rho_Q}}{\widetilde{\rho}_r \,{\rm arsh}\left(\sqrt{\frac{\rho_Q}{\widetilde{\rho}_r}}\right)}} \cdot \left(\frac{(n+2)\beta}{n|\gamma|}\right)^{(n+2)/8} e^{-N_0}. \end{aligned} \tag{3.434}$$

The only one ingredient from the second de Sitter phase is using the formula (3.428). Now we should match the solution (3.287–288) with conditions (3.90) to the solution (3.124), (3.116), (3.122). We should match a field Ψ, a density of an energy, a radius and a time. One gets

$$R_3 = \overline{R}_0 \exp\left(\frac{n+1}{M_{\rm pl}}(t_2 - \overline{t}_0)\right) = 3.07\sqrt[3]{\frac{A}{\rho_Q}} \tag{3.435}$$

$$\Psi(t_2) = \frac{1}{2n}\ln\left(\frac{\delta}{3|\gamma|}\right) - \frac{1}{n}\sqrt{\frac{2\delta}{3|\overline{M}|}}\,M_{\rm pl}(t_2 - \overline{t}_0) = \Psi_0 \tag{3.436}$$

$$\rho_Q + \frac{A}{R_3^3} = \rho_0 e^{(n+2)\Psi_0} + \rho_\Psi \tag{3.437}$$

where

$$\rho_\Psi = \frac{\delta M_{\rm pl}^2}{3n^2} - \frac{\sqrt{|\gamma|\delta}}{2\sqrt{3}}\exp\left(-\sqrt{\frac{2\delta}{3\overline{M}}}\,M_{\rm pl}(t_2 - \overline{t}_0)\right) \tag{3.438}$$

$$\delta = 2\,\frac{n+1}{M_{\rm pl}^2} - \frac{3\rho_0}{M_{\rm pl}^2} \tag{3.439}$$

$$\overline{t}_0 \neq 0. \tag{3.440}$$

One finds:

$$t_2 = \sqrt{\frac{3\overline{M}}{2\delta}}\,\frac{1}{M_{\rm pl}}\ln\left(\sqrt{\frac{\delta(n+2)^n\beta^n}{3|\gamma|^{n+1}n^n}}\right) + \overline{t}_0 \tag{3.441}$$

and t_2 is large,

$$\rho_\Psi \cong \frac{\delta M_{\rm pl}^2}{3n^2} = \frac{M_{\rm pl}^2}{3n^2}\left(2\,\frac{n+1}{M_{\rm pl}^2} - \frac{3\rho_0}{M_{\rm pl}^2}\right) \tag{3.442}$$

$$\rho_0 = \frac{n^2\lambda_{c0} - 3(n+1)}{n^2 x_0^{n+2} - 1}\,. \tag{3.443}$$

From

$$\begin{aligned} \frac{A}{R_3^3} &= -\rho_Q + \rho_0\left(x_0^{n+2} - \frac{1}{n^2}\right) + \frac{3(n+1)}{n^2} \\ R_3 &= 3.072\sqrt[3]{\frac{A}{\rho_Q}} \end{aligned} \tag{3.444}$$

one obtains

$$\delta = \frac{1}{M_{\rm pl}^2}\left(2(n+1) - 3\,\frac{\lambda_{c0}n^2 - 3(n+1)}{n^2 x_0^{n+2} - 1}\right). \tag{3.445}$$

From Eqs (3.90), (3.287) and (3.288) we get

$$t_2 = \frac{N_1}{H_1} + \frac{5.38}{\sqrt{\rho_Q}}\,M_{\rm pl} \tag{3.446}$$

$$\overline{t}_0 = \frac{N_1}{H_1} + \frac{5.38}{\sqrt{\rho_Q}}\,M_{\rm pl} - \sqrt{\frac{3\overline{M}}{2\delta}}\,\frac{1}{M_{\rm pl}}\ln\left(\sqrt{\frac{\delta(n+2)^n\beta^n}{3|\gamma|^{n+1}n^n}}\right) \tag{3.447}$$

$$M_{\rm pl}\frac{5.38}{\sqrt{\rho_Q}} = 5.64\cdot 10^{11}\ \text{yr}. \tag{3.448}$$

Finally we find $\overline{R}_0$:

$$\overline{R}_0 = R_3\exp\left(-\frac{n+1}{M_{\rm pl}}(t_2 - \overline{t}_0)\right) = 3.072\sqrt[3]{\frac{A}{\rho_Q}}\exp\left(-\frac{n+1}{M_{\rm pl}}(t_2 - \overline{t}_0)\right). \tag{3.449}$$

In this way we match an evolution of the Universe from the very beginning up to K-essence dominance. We find all the constants. Let us notice that the solution (3.90), (3.287–288) has a quite known behaviour in the theory of ordinary nonlinear differential equations. The solution cannot be extended after some value of the dependent variable (and also independent). One says the solution dies. Due to this interesting behaviour we obtain physical consequences. Matter and a quintessence cannot evolve independently. We get also a ratio between a density of matter and a quintessence which agrees with observational data and helps (in principle) to calculate a time of our contemporary epoch.

If we use Eqs (3.287) and (3.288) we can calculate a time of our contemporary epoch, i.e. a time when the ratio of a density of a matter to a quintessence density is $\frac{3}{7}$ (this is this ratio measured now). One gets

$$\overline{t}_{\text{contemporary}} = 8.97\,\frac{M_{\text{pl}}}{\sqrt{\rho_Q}} + \frac{N_1}{H_1} = 9.42\cdot 10^{10}\,\text{yr} + \frac{N_1}{H_1}\,. \tag{3.450}$$

In the same way using Eqs (3.287), (3.288) and (3.289) we calculate a Hubble parameter for our contemporary epoch.

$$H_{\text{contemporary}} = h_{\text{contemporary}}\,\frac{100\,\frac{\text{k}m}{\text{s}}}{\text{Mpc}} \tag{3.451}$$

where

$$h_{\text{contemporary}} = 1.21. \tag{3.452}$$

Both values are a little too big.

Moreover one can improve the results. However, the age of the Universe is a little longer,

$$t_{\text{age of the Universe}} = t_{\text{contemporary}} + t_1 = 9.42\cdot 10^{10}\,\text{yr} + 2\,\frac{N_1}{H_1} + \frac{N_0}{H_0}\,. \tag{3.453}$$

In general one gets

$$H = \frac{\sqrt{\rho_Q}}{M_{\text{pl}}}\,\frac{32.27\sqrt[3]{\frac{\rho_Q}{\rho_m}} - 16.38}{100\sqrt[3]{\frac{\rho_Q}{\rho_m}} - 227.48}$$

and

$$-q = \frac{5.21}{11.76\sqrt[3]{\frac{\rho_Q}{\rho_m}} - 5.17} + 1$$

where $\frac{\rho_Q}{\rho_m}$ is a ratio of a quintessence energy density to a matter (dust) energy density, or

$$h = 0.9115\,\frac{32.27\sqrt[3]{\frac{\rho_Q}{\rho_m}} - 16.38}{14.96\sqrt[3]{\frac{\rho_Q}{\rho_m}} - 0.03}\,.$$

Finally, let us express an age of the Universe in terms of the ratio $\frac{\rho_Q}{\rho_m}$. One gets

$$t\left(\frac{\rho_Q}{\rho_m}\right) = 2.667\ln\left(31.87\sqrt[3]{\frac{\rho_Q}{\rho_m}} - 14.03\right)\frac{M_{\text{pl}}}{\rho_Q} + 2\left(\frac{N_1}{H_1}\right) + \left(\frac{N_0}{H_0}\right)$$

or

$$t\left(\frac{\rho_Q}{\rho_m}\right) = 2.82\cdot 10^{10}\,\text{yr}\cdot\ln\left(31.87\sqrt[3]{\frac{\rho_Q}{\rho_m}} - 14.03\right)\frac{M_{\text{pl}}}{\rho_Q} + 2\left(\frac{N_1}{H_1}\right) + \left(\frac{N_0}{H_0}\right)\,.$$

For $\frac{\rho_m}{\rho_Q} = 0.0034$ one gets

$$t(294.12) = 14.91 \cdot 10^{10}\,\mathrm{yr} + 2\left(\frac{N_1}{H_1}\right) + \left(\frac{N_0}{H_0}\right).$$

For $\frac{\rho_m}{\rho_Q} = 2.015$

$$t(0.496) = 6.76 \cdot 10^{10}\,\mathrm{yr} + 2\left(\frac{N_1}{H_1}\right) + \left(\frac{N_0}{H_0}\right).$$

$$\Delta t = t(294.12) - t(0.496) = 8.15 \cdot 10^{10}\,\mathrm{yr}.$$

The last number is a duration time of the epoch for ρ_Q is equal to the contemporary value of a cosmological constant.

Let us come back to the Eq. (3.52) and consider it in the flat case $K = 0$. One gets

$$\frac{R\,dR}{\sqrt{\frac{\rho_Q}{3M_{\mathrm{pl}}^2} \cdot R^4 + \frac{A}{3M_{\mathrm{pl}}^2} \cdot R + \frac{B}{3M_{\mathrm{pl}}^2}}} = \pm dt. \tag{3.454}$$

We change R into x:

$$R = \sqrt[3]{\frac{A}{\rho_Q}}\,x \tag{3.455}$$

and finally get

$$\pm\int \frac{x\,dx}{\sqrt{x^4 + x + a}} = \sqrt{\frac{\rho_Q}{3M_{\mathrm{pl}}^2}}\,(t - t_0) \tag{3.456}$$

where

$$0 < a = \frac{B}{A}\sqrt[3]{\frac{\rho_Q}{A}}. \tag{3.457}$$

This is an evolution of a flat Universe with radiation, matter (dust) and a density of a quintessence (this is a cosmological constant λ_{c0}). The integral on LHS of Eq. (3.456) is an elliptic integral and can be calculated. The properties of the result of calculations strongly depend on the value of a. Let us notice that the integral we have calculated here has $a = 0$ (no radiation).

The evolution of a quintessence, a matter and a radiation is here independent. One gets the following results for

$$I = \int \frac{x\,dx}{\sqrt{x^4 + x + a}}. \tag{3.458}$$

In general there are two cases:

I $$B_2 > 0 \tag{3.459}$$

II $$B_2 < 0 \tag{3.460}$$

where

$$B_2 = \frac{\sqrt{12b_1^4 - a} - 2b_1^2 - \sqrt{4b_1^4 - a}}{2\sqrt{12b_1^4 - a}} \tag{3.461}$$

and $b_1 > 0$ is a solution of the cubic equation

$$y^3 - ay - \tfrac{1}{8} = 0 \tag{3.462}$$

$$y = 2b_1^4 \tag{3.463}$$

Eq. (3.462) can be solved by using the Cardano formulae in two cases
1) $D > 0$,
2) $D < 0$,

$$D = \frac{27 - 256a^3}{27 \cdot 256}\,. \tag{3.464}$$

In case 1) one gets

$$\begin{aligned} b_1 &= \frac{1}{2}\sqrt[4]{\sqrt[3]{4 + \tfrac{1}{2}\sqrt{27 - 256a^3}} + \sqrt[3]{4 - \tfrac{1}{2}\sqrt{27 - 256a^3}}} > 0 \\ 0 &< a < \frac{3}{4\sqrt[3]{4}} = 0.472470393\ldots \end{aligned} \tag{3.465}$$

In case 2) one gets

$$b_1 = \sqrt[4]{\frac{a}{3}}\sqrt{\cos\frac{\varphi}{3}} \tag{3.466}$$

where

$$\cos\varphi = \frac{3}{16a}\sqrt{\frac{3}{a}} \tag{3.467}$$

$$a > \frac{3}{4\sqrt[3]{4}}\,, \quad \varphi \in \langle 0, \tfrac{\pi}{2}\rangle. \tag{3.468}$$

Condition I can be transformed into

$$4b_1^2 < a + 1 \tag{3.469}$$

for both cases 1) and 2). In case 1) this condition has no solution. It means we have always $B_2 < 0$. In case 2) we have always (3.469). It means

$$B_2 > 0 \tag{3.470}$$

for an equation

$$\frac{1}{\sqrt[4]{4}}\,\frac{\cos^2\left(\frac{\varphi}{3}\right)}{\cos^{4/3}\varphi} = \frac{3}{4\sqrt[3]{4}\cos^{2/3}\varphi} + 1. \tag{3.471}$$

has no solution in $(0, \frac{\pi}{2})$ and in $(\frac{3\pi}{2}, 2\pi)$.

The fact that in case 1) we have always

$$4b_1^2 > a + 1 \tag{3.472}$$

is caused by a nonexistence of real solutions of an equation

$$4(a+1)^{1/2} = \sqrt[3]{4 + \tfrac{1}{2}\sqrt{27 - 256a^3}} + \sqrt[3]{4 - \tfrac{1}{2}\sqrt{27 - 256a^3}} \tag{3.473}$$

where

$$0 < a < \frac{3}{4\sqrt[3]{4}}\,. \tag{3.474}$$

Thus for sufficiently small a we always have case I (in a limit $a = 0$, no radiation also).

One can express b_1 and a in terms of φ:

$$b_1^4 = \frac{3}{4\sqrt[3]{4}}\,\frac{\cos^2(\frac{\varphi}{3})}{\cos^{2/3}\varphi} \tag{3.475}$$

$$a = \frac{3}{4\sqrt[3]{4}}\cos^{-2/3}\varphi. \tag{3.476}$$

Thus one obtains in case I

$$\begin{aligned} I &= \frac{1}{2}\ln\frac{e(\frac{x-\alpha}{x-\beta})^2 + f + 2\sqrt{\big((\frac{x-\alpha}{x-\beta})^2 + c\big)\big((\frac{x-\alpha}{x-\beta})^2 + d\big)(1+c)(1+d)}}{(\frac{x-\alpha}{x-\beta})^2 - 1} \\ &\quad + K_1(b_1,a)\varPi\left(\operatorname{arctg}\left[P(b_1,a)\,\frac{x-\alpha}{x-\beta}\right], n_1, q_1\right) \\ &\quad + L_1(b_1,a)F\left(\operatorname{arctg}\left[P(b_1,a)\,\frac{x-\alpha}{x-\beta}\right], q_1\right) \end{aligned} \tag{3.477}$$

$$P_1(b_1,a) = \left(\frac{\sqrt{12b_1^4 - a} - 2b_1^2 - \sqrt{4b_1^4 - a}}{\sqrt{12b_1^4 - a} + 2b_1^2 + \sqrt{4b_1^4 - a}}\right)^{1/2} \tag{3.478}$$

$$q_1 = \frac{\sqrt{6}|b_1|}{\left(3b_1^2 + \sqrt{12b_1^4 - a}\right)^{1/2}} \tag{3.479}$$

$$K_1(b_1,a) = \left(\frac{2b_1^2 + \sqrt{4b_1^4 - a} + \sqrt{12b_1^4 - a}}{\sqrt{12b_1^4 - a} - 2b_1^2 - \sqrt{4b_1^4 - a}}\right)^{3/2} \tag{3.480}$$

$$n_1 = \frac{2\sqrt{12b_1^4 - a}}{2b_1^2 + \sqrt{4b_1^4 - a} + \sqrt{12b_1^4 - a}} \tag{3.481}$$

$$L_1(b_1,a) = \left(M(b_1,a) + \frac{2b_1^2 + \sqrt{4b_1^4 - a} + \sqrt{12b_1^4 - a}}{2\sqrt{12b_1^4 - a}}\right) \frac{1}{\left(3b_1^2 + \sqrt{12b_1^4 - a}\right)^{1/2}}$$

$$\times \frac{2(12b_1^4 - a)^{1/2}\left(\sqrt{4b_1^4 - a} - b_1^2\right)^{1/2}}{\left(2b_1^2 + \sqrt{4b_1^4 - a} - \sqrt{12b_1^4 - a}\right)\left(2b_1^2 + \sqrt{4b_1^4 - a} + \sqrt{12b_1^4 - a}\right)^{1/2}} \qquad (3.482)$$

$$\begin{aligned}
M(b_1,a) &= \frac{\beta}{\alpha - \beta} \\
&= \Bigg((6b_1^4 + a)\sqrt{4b_1^4 - a} + 2b_1^2 a + \sqrt{192b_1^{12} - 112b_1^8 a + 20b_1^4 a + a^3} \\
&\quad + \sqrt{576b_1^{12} - 240b_1^8 a + 28b_1^4 a - a^3} + 6b_1^2\sqrt{48b_1^8 - 16b_1^4 a + a^2} \\
&\quad + (a - 4b_1^4)\sqrt{12b_1^4 - a}\Bigg)^{1/2} \\
&\quad \times \Bigg(\bigg((12b_1^4 - a)\sqrt{4b_1^4 - a} + 4b_1^4\sqrt{12b_1^4 - a} - 4b_1^2 a + 24b_1^6 + 4b_1^4\sqrt{12b_1^4 - a} \\
&\qquad - \sqrt{192b_1^{12} - 112b_1^8 a + 20b_1^4 a + a^3} - \sqrt{576b_1^{12} - 240b_1^8 a + 28b_1^4 a - a^3}\bigg)^{1/2} \\
&\quad - \bigg(8b_1^2\sqrt{48b_1^8 - 16b_1^4 a + a^2} + 2(6b_1^4 - a)\sqrt{4b_1^4 - a} - 4b_1^4\sqrt{12b_1^4 - a} \\
&\qquad + 4b_1^6 + 6b_1^2 a + \sqrt{192b_1^{12} - 112b_1^8 a + 20b_1^4 a + a^3} \\
&\qquad + \sqrt{576b_1^{12} - 240b_1^8 a + 28b_1^4 a - a^3}\bigg)^{1/2}\Bigg)^{-1}. \qquad (3.483)
\end{aligned}$$

$$\alpha = \sqrt{\frac{\sqrt{48b_1^8 - 16b_1^4 a + a^2} + 6b_1^2\sqrt{4b_1^4 - a} + 2b_1^2\sqrt{12b_1^4 - a} + a}{2b_1^2 + \sqrt{4b_1^4 - a} + \sqrt{12b_1^4 - a}}} > 0 \qquad (3.484)$$

$$\beta = \sqrt{\frac{\sqrt{48b_1^8 - 16b_1^4 a + a^2} + 6b_1^2\sqrt{4b_1^4 - a} - 2b_1^2\sqrt{12b_1^4 - a} + a}{2b_1^2 - \sqrt{4b_1^4 - a} - \sqrt{12b_1^4 - a}}} > 0 \qquad (3.485)$$

$$c = \frac{\left(3b_1^2 + \sqrt{12b_1^4 - a}\right)\left(2b_1^2 - \sqrt{4b_1^4 - a} - \sqrt{12b_1^4 - a}\right)}{\left(\sqrt{4b_1^4 - a} + 2b_1^2\right)\left(\sqrt{12b_1^4 - a} - 3b_1^2\right)}$$

$$d = \frac{\sqrt{12b_1^4 - a} + 2b_1^2 + \sqrt{4b_1^4 - a}}{\sqrt{12b_1^4 - a} - 2b_1^2 - \sqrt{4b_1^4 - a}}, \quad |d| > 1 \qquad (3.486)$$

$$e = \frac{2\sqrt{12b_1^4 - a}\left(\sqrt{12b_1^4 - a} + \sqrt{4b_1^4 - a} - 4b_1^2\right)}{\left(\sqrt{4b_1^4 - a} + 2b_1^2 + \sqrt{12b_1^4 - a}\right)\left(\sqrt{12b_1^4 - a} - 3b_1^2\right)} \tag{3.487}$$

$$\begin{aligned} f = {} & \frac{2\sqrt{12b_1^4 - a}}{\left(\sqrt{12b_1^4 - a} - 3b_1^2\right)\left(\sqrt{12b_1^4 - a} - 2b_1^2 - \sqrt{4b_1^4 - a}\right)} \\ & \times \left(2b_1^2 + \sqrt{4b_1^4 - a} + \sqrt{12b_1^4 - a}\right)^{-2} \\ & \times \Big(2(14b_1^4 - a)\sqrt{4b_1^4 - a} - 4b_1^2 a + 2\sqrt{192b_1^{12} - 112b_1^8 a + 20b_1^4 a + a^3} \\ & \quad + 2(b_1^4 - a)\sqrt{12b_1^4 - a} - 2\sqrt{576b_1^{12} - 240b_1^8 a + 28b_1^4 a - a^3}\Big). \end{aligned} \tag{3.488}$$

Case II

$$\begin{aligned} I = {} & \frac{1}{2}\ln\frac{e(\frac{x-\alpha}{x-\beta})^2 + f + 2\sqrt{\left((\frac{x-\alpha}{x-\beta})^2 + c\right)\left((\frac{x-\alpha}{x-\beta})^2 + d\right)(1+c)(1+d)}}{(\frac{x-\alpha}{x-\beta})^2 - 1} \\ & + K_2(b_1, a)\mathit{\Pi}\left(\arccos\left[P_2(b_1, a)\,\frac{x-\alpha}{x-\beta}\right], n_2, q_2\right) \\ & + L_2(b_1, a)F\left(\arccos\left[P_2(b_1, a)\,\frac{x-\alpha}{x-\beta}\right], q_2\right) \end{aligned} \tag{3.489}$$

$$q_2 = \frac{1}{q_1} = \frac{\left(3b_1^2 + \sqrt{12b_1^4 - a}\right)^{1/2}}{\sqrt{6}|b_1|} \tag{3.490}$$

$$n_2 = \frac{2b_1^2 + \sqrt{4b_1^4 - a} + \sqrt{12b_1^4 - a}}{2\sqrt{12b_1^4 - a}} \tag{3.491}$$

$$K_2(b_1, a) = \frac{\left(2b_1^2 + \sqrt{4b_1^4 - a} - \sqrt{12b_1^4 - a}\right)\sqrt{12b_1^4 - a}\left(\sqrt{4b_1^4 - a} - b_1\right)^{1/2}}{\left(\sqrt{12b_1^4 - a} + 2b_1^2 + \sqrt{4b_1^4 - a}\right)\left(\sqrt{12b_1^4 - a} - 3b_1^2\right)^{3/2}} \tag{3.492}$$

$$P_2(b_1, a) = \left(\frac{2b_1^2 + \sqrt{4b_1^4 - a} - \sqrt{12b_1^4 - a}}{\sqrt{12b_1^4 - a} + 2b_1^2 + \sqrt{4b_1^4 - a}}\right)^{1/2} \tag{3.493}$$

$$\begin{aligned} & L_2(b_1, a) = M(b_1, a) \\ & \quad - \frac{4(12b_1^4 - a)\left(\sqrt{4b_1^4 - a} - b_1^2\right)^{1/2}\left(\sqrt{12b_1^4 - a} - 2b_1^2 - \sqrt{4b_1^4 - a}\right)}{\left(3b_1^2 - \sqrt{12b_1^4 - a}\right)^{3/2}\left(2b_1^2 + \sqrt{4b_1^4 - a} + \sqrt{12b_1^4 - a}\right)}. \end{aligned} \tag{3.494}$$

The function (3.456) cannot be inverted globally. It can be inverted only locally in some intervals where the solution lives. The solution dies on the end of an interval being reborning on beginning of a next interval. All the intervals can be obtained from the condition

$$\overline{W} > 0 \tag{3.495}$$

where

$$\overline{W} = \frac{e(\frac{x-\alpha}{x-\beta})^2 + f + 2\sqrt{\big((\frac{x-\alpha}{x-\beta})^2 + c\big)\big((\frac{x-\alpha}{x-\beta})^2 + d\big)(1+c)(1+d)}}{(\frac{x-\alpha}{x-\beta})^2 - 1} \tag{3.496}$$

and

$$\left|\frac{x-\alpha}{x-\beta}\right| \neq 1 \tag{3.497}$$

In the case I $c > 0$, $d > 0$.

In the case II $c > 0$, $d < 0$.

Thus one can get very interesting behaviour from the physical point of view, because every end of an interval of living (existence) of the solution of an evolution equation means a start of nontrivial interaction between a matter, a radiation and a quintessence. In all of these intervals one can find an approximate inverse function, i.e. one can write

$$R = F(t) \tag{3.498}$$

where

$$F(t_1) = R_1 < R < R_2 = F(t_2) \tag{3.499}$$

and

$$t_1 < t < t_2. \tag{3.500}$$

Moreover we do not develop this project here in details. F and Π are usual elliptic integrals of the first and of the third kind given by the formulae (3.71) and (3.73).

For future convenience of this project we find roots of the polynomial

$$P(x) = x^4 + x + a, \quad a > 0. \tag{3.501}$$

First of all according to the Ferrari method we should solve a resolvent equation which is a cubic equation. One gets

$$z^3 - 4az - 1 = 0. \tag{3.502}$$

The discriminant of Eq. (3.502) reads

$$\overline{D} = \frac{27 - 256a^3}{108}$$

and we have two cases

$$\overline{D} > 0, \quad 0 < a < \frac{3}{4\sqrt[3]{4}}, \tag{3.503}$$

$$\overline{D} < 0, \quad a > \frac{3}{4\sqrt[3]{4}}, \tag{3.504}$$

In the case (3.503) one gets

$$z_1 = \sqrt[3]{\frac{\sqrt{27-256a^3}+3\sqrt{3}}{6\sqrt{3}}} - \sqrt[3]{\frac{\sqrt{27-256a^3}-3\sqrt{3}}{6\sqrt{3}}} \tag{3.505}$$

$$z_{2,3} = \frac{1}{2}\left(\sqrt[3]{\frac{\sqrt{27-256a^3}-3\sqrt{3}}{6\sqrt{3}}} - \sqrt[3]{\frac{\sqrt{27-256a^3}+3\sqrt{3}}{6\sqrt{3}}}\right)$$
$$\pm \frac{i\sqrt{3}}{2}\left(\sqrt[3]{\frac{\sqrt{27-256a^3}-3\sqrt{3}}{6\sqrt{3}}} + \sqrt[3]{\frac{\sqrt{27-256a^3}+3\sqrt{3}}{6\sqrt{3}}}\right) \tag{3.506}$$

In the second case (3.504)

$$z_1 = 4\sqrt{\frac{a}{3}}\cos\left(\frac{\overline{\varphi}}{3}\right) \tag{3.507}$$

$$z_2 = 4\sqrt{\frac{a}{3}}\cos\left(\frac{\overline{\varphi}}{3}+\frac{2\pi}{3}\right) \tag{3.508}$$

$$z_3 = 4\sqrt{\frac{a}{3}}\cos\left(\frac{\overline{\varphi}}{3}+\frac{4\pi}{3}\right) \tag{3.509}$$

where $\cos\overline{\varphi} = \sqrt{\frac{3}{16a}}$.

We can also distinguish the special case $\overline{D} = 0$ ($a = \frac{3}{4\sqrt[3]{4}}$):

$$z_1 = \sqrt[3]{4} \tag{3.510}$$

$$z_{2,3} = -\tfrac{1}{2}\sqrt[3]{4} \tag{3.511}$$

In the first case we get one real and two complex conjugated roots, in the second three real roots, in the third case two different real roots (one of them is double).

According to the Ferrari method one gets

$$2x_1 = \sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3} \tag{3.512a}$$

$$2x_2 = \sqrt{z_1}-\sqrt{z_2}-\sqrt{z_3} \tag{3.512b}$$

$$2x_3 = -\sqrt{z_1}+\sqrt{z_2}-\sqrt{z_3} \tag{3.512c}$$

$$2x_4 = -\sqrt{z_1}+\sqrt{z_2}+\sqrt{z_3} \tag{3.512d}$$

$$z_1 z_2 z_3 = 1. \tag{3.513}$$

In the future analysis of an evolution of our model of the Universe there are three important cases for z_1, z_2 and z_3:

A. All z_1, z_2, z_3 are real and positive. In this case all x_1, x_2, x_3 and x_4 are real.

B. One root (e.g. z_1) is positive, two remaining (z_2, z_3) are negative. In this case we have two pairs of conjugate roots.

C. One root (e.g. z_1) is positive, two remaining (z_2, z_3) are complex conjugate. In this case we have two real roots and one pair of complex conjugate roots.

This analysis is quite important because only for positive real roots the integral I can have singular points. It means in A and C case. This gives us some restrictions on the parameter a.

Let us notice that in the case of $a = 0$ (considered before) we have to do with the case C. Fortunately one real root is zero and the second real root is negative ($x_2 = -1$).

Let us notice that our case with $a = 0$ is in some sense exceptional from the point of view of a general theory. In that case some of the formulae are singular and because of this it was considered separately.

Let us notice that for sufficiently big a the polynomial (501) has no real roots. This is true for $a > \frac{3}{4\sqrt[3]{4}}$. In this case the integral (458) has no singular points. This is case II.

Conclusions

In the paper we consider some cosmological consequences of the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory. Especially we use the scalar field Ψ appearing there in order to get cosmological models with a quintessence and phase transitions. We consider a dynamics of Higgs' fields with various approximations and models of inflation [14].

Eventually we develop a toy model of this dynamics to obtain an amount of inflation and $P_R(K)$ function (a spectral function for fluctuations). We calculate a spectral index $n_s(K)$ and $\dfrac{dn_s}{d\ln K}$ for this model. We calculate a Hubble parameter and an age of the Universe.

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